Deformed algebra and the effective dynamics of the interior of black holes

Wilfredo Yupanqui Carpio

División de Ciencias e Ingenierías-Universidad de Guanajuato
Campus León

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Introduction

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Introduction

The standard expression of the metric of Schwarzschild black hole in terms of spherical coordinates is given by

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \quad (1)$$

this is a vacuum solution from Einstein field equations. The black hole singularity corresponds to $r = 0$ and the event horizon corresponds to $r = 2GM$. Inside the horizon, the temporal and radial coordinates flip roles, then, the metric of the Schwarzschild interior now reads as the form of

$$ds^2 = -\left(\frac{2GM}{t} - 1\right)^{-1} dt^2 + \left(\frac{2GM}{t} - 1\right) dx^2 + t^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right), \quad (2)$$

with $t \in (0, 2GM)$ and $r \in (-\infty, \infty)$, and M is the mass of the black hole. Here, the metric coefficients become infinite at $t = 0$ and $t = 2GM$. In Schwarzschild metric, an invariant used to find the true singularities of a spacetime is the Kretschmann scalar

$$K = R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} = \frac{48G^2M^2}{t^6} \quad (3)$$

At $t = 0$ the Kretschmann scalar diverges, then, this point represent an honest singularity of spacetime. In $t = 2GM$ none of the curvature invariants blows up there, then, this point is not a singularity.

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The Schwarzschild interior metric can be written in terms of Ashtekar variables:

$$ds^2 = -N^2 dt^2 + \frac{p_b^2}{L_0^2 |p_c|} dx^2 + |p_c| (d\theta^2 + \sin^2\theta d\phi^2) \tag{4}$$

This metric is a special case of a Kantowski-Sachs cosmological spacetime. From here, we conclude that

$$\frac{p_b^2}{L_0^2 |p_c|} = \left(\frac{2GM}{t} - 1 \right), \qquad |p_c| = t^2. \tag{5}$$

This means that

$$p_b = 0, \qquad p_c = 4G^2 M^2, \qquad \text{On the horizon } t = 2GM, \tag{6}$$

$$p_b \to 0, \qquad p_c \to 0, \qquad \text{At singularity } t = 0, \tag{7}$$

and we have used the Schwarzschild lapse $N = \left(\frac{2GM}{t} - 1 \right)^{-\frac{1}{2}}$. In these variables the Kretschammn scalar becomes

$$K = \frac{12 \left(b^2 + \gamma^2 \right)^2}{\gamma^4 p_c^2} \tag{8}$$

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Classical solutions

The full Hamiltonian of gravity written in Ashtekar connection variables is

$$H = -\frac{N\mathrm{sgn}(p_c)}{2G\gamma^2}\left[2bc\sqrt{|p_c|} + \left(b^2+\gamma^2\right)\frac{p_b}{\sqrt{|p_c|}}\right], \tag{9}$$

here b and c are the connections and its conjugate momenta, p_b and p_c), which are the densitized triad. Also, γ is the Barbero-Immirzi parameter. The Poisson brackets between these variables are

$$\{c, p_c\} = 2G\gamma, \qquad \{b, p_b\} = G\gamma. \tag{10}$$

In order to be able to compare the effects of GUP, we first need to have the classical dynamics. Choosing a lapse

$$N(T) = \frac{\gamma\,\mathrm{sgn}(p_c)\sqrt{|p_c(T)|}}{b(T)} \tag{11}$$

the Hamiltonian constraint (9) becomes

$$H = -\frac{1}{2G\gamma}\left[\left(b^2+\gamma^2\right)\frac{p_b}{b} + 2cp_c\right]. \tag{12}$$

The reason for choosing the lapse (11) is that as we will see, the equations of motion of the pair (c, p_c) decouple from those of (b, p_b).

The classical Hamiltonian

The equations of motion corresponding to (12) are

$$\frac{db}{dT} = \{b, H\} = -\frac{1}{2}\left(b + \frac{\gamma^2}{b}\right), \tag{13}$$

$$\frac{dp_b}{dT} = \{p_b, H\} = \frac{p_b}{2}\left(1 - \frac{\gamma^2}{b^2}\right). \tag{14}$$

$$\frac{dc}{dT} = \{c, H\} = -2c, \tag{15}$$

$$\frac{dp_c}{dT} = \{p_c, H\} = 2p_c, \tag{16}$$

These equations should be supplemented by the weakly vanishing (≈ 0) of the Hamiltonian constraint (12),

$$\left(b^2 + \gamma^2\right)\frac{p_b}{b} + 2cp_c \approx 0. \tag{17}$$

It is clear that p_c is the square of the radius of the infalling 2-spheres.

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Thus, the solutions to the equations of motion in terms of Schwarzschild time t become

$$b(t) = \pm\gamma\sqrt{\frac{2GM}{t} - 1}, \tag{18}$$

$$p_b(t) = lL_0 t\sqrt{\frac{2GM}{t} - 1}, \tag{19}$$

$$c(t) = \mp\frac{\gamma GMlL_0}{t^2}, \tag{20}$$

$$p_c(t) = t^2. \tag{21}$$

The behavior of these solutions as a function of t is depicted in Fig. 1. From these equations or the plot, one can see that $p_c \to 0$ as $t \to 0$, i.e., at the classical singularity, leading to the Riemann invariants such as the Kretschmann scalar

$$K = \frac{12\left(b^2 + \gamma^2\right)^2}{\gamma^4 p_c^2}$$

all diverge, signaling the presence of a physical singularity there as expected.

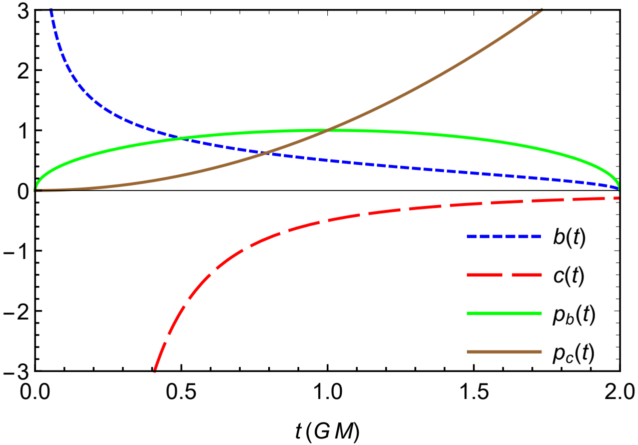

Figure: The behavior of canonical variables as a function of Schwarzschild time t. We have chosen the positive sign for b and negative sign for c. The figure is plotted using $\gamma = 0.5,\ M = 1,\ G = 1$ and $L_0 = 1$.

Effective dynamics inspired by Generalized Uncertainty Principle (GUP)

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GUP

In one dimension the simplest generalized uncertainty relation which implies the appearance of a nonzero minimal uncertainty Δx in position has the form

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left[1 + \beta\left(\Delta p\right)^2 + \gamma\right] \tag{22}$$

where β and γ are positive and independent of Δx and Δp (but may in general depend on the expectation values of x and p).

Now in general it is known that for any pair of observables A, B which are represented as symmetric operators on a domain of A and B the uncertainty relation

$$\Delta A \Delta B \geq \frac{\hbar}{2}\left|\langle[A, B]\rangle\right| \tag{23}$$

We consider the associative Heisenberg algebra generated by x and p obeying the commutation relation ($\beta > 0$)

$$[\mathbf{x}, \mathbf{p}] = i\hbar\left(1 + \beta\mathbf{p}^2\right). \tag{24}$$

The corresponding uncertainty relation is

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$$\Delta x \Delta p \geq \frac{\hbar}{2}\left[1+\beta\left(\Delta p\right)^2+\beta\langle \mathbf{p}\rangle^{\mathbf{2}}\right]. \tag{25}$$

From here

$$\Delta p = \frac{\Delta x}{\hbar\beta} \pm \sqrt{\left(\frac{\Delta x}{\hbar\beta}\right)^2 - \frac{1}{\beta} - \langle \mathbf{p}\rangle^{\mathbf{2}}}. \tag{26}$$

One reads off the minimal position uncertainty

$$\Delta x_{min}\left(\langle \mathbf{p}\rangle\right) = \hbar\sqrt{\beta}\sqrt{1+\beta\langle \mathbf{p}\rangle^{\mathbf{2}}} \tag{27}$$

so that the absolutely smallest uncertainty in positions has the value

$$\Delta x_0 = \hbar\sqrt{\beta}. \tag{28}$$

There is no nonvanishing minimal uncertainty in momentum.

Deformation of Poisson brackets

In order to find the effective GUP-modified dynamics, we impose a minimal uncertainty in p_b and p_c, and thus we modify the classical algebra of variables. To be as general as possible, let us call the configuration variables q_1, q_2 and the momenta p_1, p_2. In our case

$$q_1 = \frac{b}{\gamma}, \qquad q_2 = \frac{c}{\gamma}, \tag{29}$$

$$p_1 = \frac{1}{G} p_b, \qquad p_2 = \frac{1}{2G} p_c. \tag{30}$$

Thus, we modify the algebra such that

$$\{q_1, p_1\}_{\bar{q},p} = f(q_1, q_2), \tag{31}$$

$$\{q_2, p_2\}_{\bar{q},p} = g(q_1, q_2), \tag{32}$$

Our purpose here is to impose alternative relations to (10) in order to reproduce GUP effects. For this reason, it is convenient to define a new pair of configuration variables conjugate to p_i. That is, we introduce the quantities $\bar{q}_1$ and $\bar{q}_2$ such that

$$\{\bar{q}_1, p_1\} = 1, \tag{33}$$

$$\{\bar{q}_2, p_2\} = 1. \tag{34}$$

This means that pairs (q_1, p_1), as well as (q_2, p_2), are no longer canonically conjugate. Note that the Poisson brackets are evaluated with respect to $\bar{q}_i, p_i$. The quantities $\bar{q}_i$ can then be constructed starting from (31) and (32), that is

$$\frac{\partial q_1}{\partial \bar{q}_1} = f\left(q_1, p_1\right), \tag{35}$$

$$\frac{\partial q_2}{\partial \bar{q}_2} = g\left(q_2, p_2\right), \tag{36}$$

whence

$$\bar{q}_1 = \int_{q_{1(0)}}^{q_1} \frac{dq_1'}{f\left(q_1', p_1\right)}, \tag{37}$$

$$\bar{q}_2 = \int_{q_{2(0)}}^{q_2} \frac{dq_2'}{g\left(q_2', p_2\right).} \tag{38}$$

In this work, we consider functions f, g such that

$$f\left(q_1, q_2\right) = f\left(q_1\right) = 1 + \beta_1 q_1^2, \tag{39}$$

$$g\left(q_1, q_2\right) = g\left(q_2\right) = 1 + \beta_2 q_2^2. \tag{40}$$

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where β_1 and β_2 are suitable dimensional parameters. In other words, we assume that any effects due to a minimal uncertainty in p_1 does not influence p_2 and vice versa. Using the specific variables of our model (b, p_b, c, p_c), the algebra (31)-(32) becomes

$$\{b, p_b\} = G\gamma\left(1 + \beta_b b^2\right), \tag{41}$$

$$\{c, p_c\} = 2G\gamma\left(1 + \beta_c c^2\right), \tag{42}$$

where we have renamed $\beta_1 \to \beta_b$ and $\beta_2 \to \beta_c$. When the modified algebra above is regarded in a quantum context, it implies a minimal uncertainty in p_b and p_c. In fact, considering the corresponding commutation relations

$$[b, p_b] = iG\gamma\left(1 + \beta_b b^2\right), \tag{43}$$

$$[c, p_c] = i2G\gamma\left(1 + \beta_c c^2\right), \tag{44}$$

one can find the following uncertainty relations

$$\Delta b \Delta p_b \geq \frac{G\gamma}{2}\left[1 + \beta_b(\Delta b)^2\right], \tag{45}$$

$$\Delta c \Delta p_c \geq G\gamma\left[1 + \beta_c(\Delta c)^2\right], \tag{46}$$

which correspond to minimal uncertainties for p_b and p_c of the order of $G\gamma\sqrt{\beta_b}$ and $2G\gamma\sqrt{\beta_c}$, respectively. Therefore, β_b and β_c effectively define the magnitude of the effects introduced with the algebra (47)-(47).

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Modified dynamics

In order to find the effective GUP-modified dynamics, we impose a minimal uncertainty in p_b and p_c, and thus we modify the classical algebra of variables

$$\{q_1, p_1\} = \left(1 + \beta_1 q_1^2\right), \qquad \{q_2, p_2\} = \left(1 + \beta_2 q_2^2\right), \tag{47}$$

Using this new algebra and the Hamiltonian (12), the new GUP-modified equations of motion become

$$\frac{db}{dT} = \{b, H\} = -\frac{1}{2}\left(b + \frac{\gamma^2}{b}\right)\left(1 + \beta_b b^2\right), \tag{48}$$

$$\frac{dp_b}{dT} = \{p_b, H\} = \frac{p_b}{2}\left(1 - \frac{\gamma^2}{b^2}\right)\left(1 + \beta_b b^2\right), \tag{49}$$

$$\frac{dc}{dT} = \{c, H\} = -2c\left(1 + \beta_c c^2\right), \tag{50}$$

$$\frac{dp_c}{dT} = \{b, H\} = 2p_c\left(1 + \beta_c c^2\right). \tag{51}$$

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$$b(t) = \pm \frac{\gamma\sqrt{2GMt^{\beta_b\gamma^2} - t\left(2\gamma^2 GM\right)^{\beta_b\gamma^2}}}{\sqrt{t\left(2\gamma^2 GM\right)^{\beta_b\gamma^2} - 2\beta_b\gamma^2 GMt^{\beta_b\gamma^2}}}, \tag{52}$$

$$p_b(t) = \frac{\ell_c}{\sqrt{-\beta_c}} t^{-\beta_b\gamma^2} \sqrt{\left[2GMt^{\beta_b\gamma^2} - t\left(2\gamma^2 GM\right)^{\beta_b\gamma^2}\right]\left[t\left(2\gamma^2 GM\right)^{\beta_b\gamma^2} - 2\beta_b\gamma^2 GMt^{\beta_b\gamma^2}\right]}, \tag{53}$$

$$c(t) = \mp \frac{\ell_c}{\sqrt{-\beta_c}} \frac{\gamma GM}{\sqrt{t^4 + \ell_c^2\gamma^2 G^2 M^2}}, \tag{54}$$

$$p_c(t) = \sqrt{t^4 + \ell_c^2\gamma^2 G^2 M^2}. \tag{55}$$

The behavior of these solutions is depicted in figures below.

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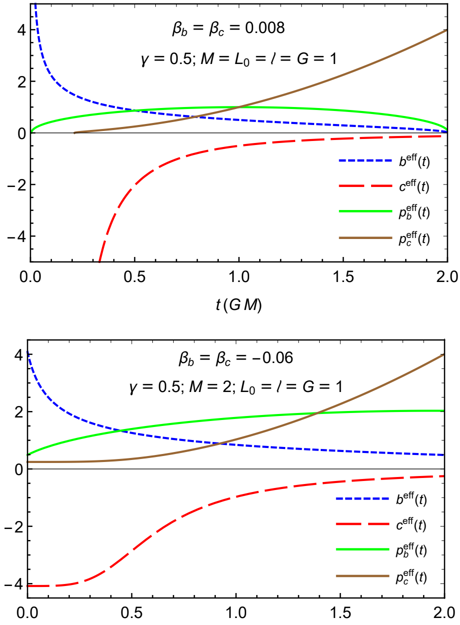
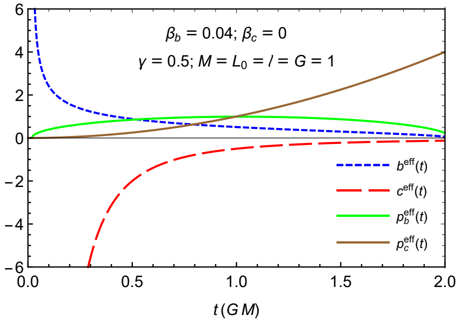
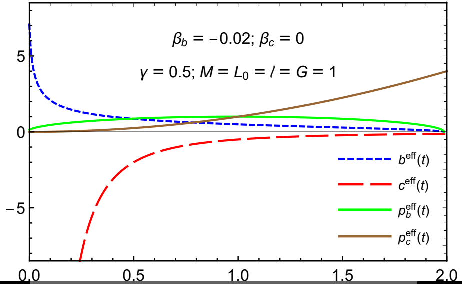

$\beta_b = \beta_c = 0.008$
$\gamma = 0.5$; $M = L_0 = l = G = 1$
$b^{\rm eff}(t)$
$c^{\rm eff}(t)$
$p_b^{\rm eff}(t)$
$p_c^{\rm eff}(t)$
$t\,(G\,M)$
$\beta_b = \beta_c = -0.06$
$\gamma = 0.5$; $M = 2$; $L_0 = l = G = 1$
$b^{\rm eff}(t)$
$c^{\rm eff}(t)$
$p_b^{\rm eff}(t)$
$p_c^{\rm eff}(t)$





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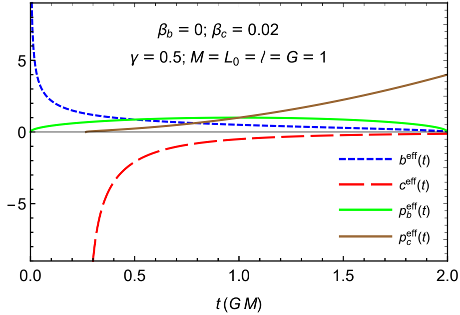

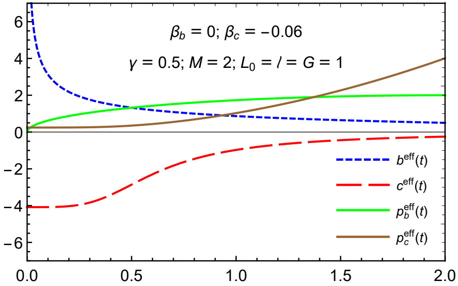

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Minimum value of p_c, comparison to LQG, and the value of ℓ_c

From (55), one can see that the minimum value of p_c happens at $t = 0$ for which p_c becomes

$$p_c^{\text{min-GUP}} = \gamma G M \ell_c. \tag{56}$$

It is seen that two free parameters contribute to such a minimum value: the LQG Barbero-Immirzi parameter γ and the GUP minimal length scale ℓ_c. Hence the existence of such a minimum p_c is purely quantum gravitational, due to the dependence on the mentioned parameters. We can go further and compare this minimum value with the value derived in the framework of LQG. There, the minimum value of p_c for LQG was found to be

$$p_c^{\text{min-LQG}} = \gamma G M \sqrt{\Delta}, \tag{57}$$

where Δ is the minimum of the area in LQG which is proportional to the Planck length squared ℓ_p^2. If one identifies $p_c^{\text{min-GUP}} = p_c^{\text{min-LQG}}$, then one would obtain

$$\ell_c^2 = \Delta. \tag{58}$$

Assuming such an identification, one can even go further and derive a relation between β_c and μ_c, the polymer parameter associated to the radial direction in LQG.

$$\mu_c = \frac{\sqrt{\Delta}}{L_0}, \tag{59}$$

one can deduce

$$-\beta_c = \mu_c^2. \tag{60}$$

The above value of $\ell_c^2 \propto \ell_p^2$ is quite small as expected and consequently one can conclude that the effective corrections kick in very close to the singularity and become dominant in that region, while for the majority of the interior the behavior mimics the classical solutions.

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Modification to the behavior at the horizon

Considering the classical solutions and, the classical values of the canonical variables at the horizon are,

$$b = 0, \quad p_b = 0, \tag{61}$$

$$c = \mp \frac{\ell_c}{\sqrt{-\beta_c}} \frac{\gamma}{4GM}, \quad p_c = 4G^2M^2. \tag{62}$$

The effective solutions, however, take on modified values at the horizon. To first order of Taylor expansion in β_i, these values are

$$b = \pm \sqrt{\beta_b}\gamma^2\sqrt{-2\ln(\gamma)}, \quad p_b = 2GM\gamma\ell_c\sqrt{2\frac{\beta_b}{\beta_c}\ln(\gamma)}, \tag{63}$$

$$c = \mp \frac{\ell_c}{\sqrt{-\beta_c}} \frac{\gamma}{4GM}\left(1 - \frac{\gamma^2\ell_c^2}{32G^2M^2}\right), \quad p_c = 4G^2M^2 - \frac{\gamma^2}{8}\ell_c^2. \tag{64}$$

From here one can see the effective model we introduced also affects the dynamics near the horizon, albeit to a very small degree, given the shear small values of ℓ_c being proportional to the Planck length. It is clear from the expressions above that the modifications to the horizon is not only affected by ℓ_c but also by the Barbero-Immirzi parameter γ.

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Final comments

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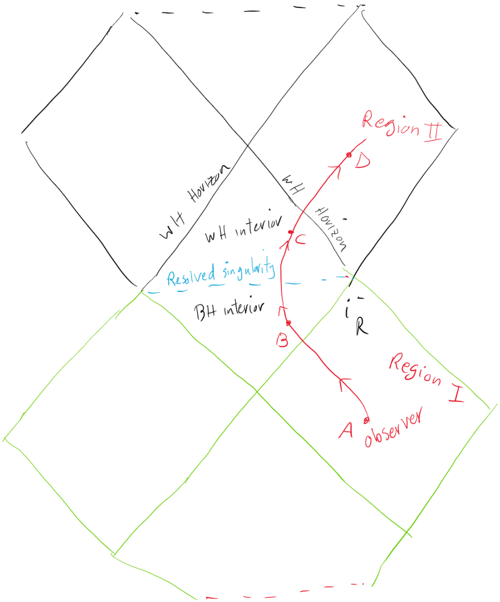

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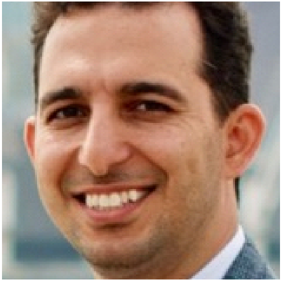

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Figure: Pasquale Bosso, Saeed Rastgoo, Octavio Obregón.


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