Non-extensive entropies that depend only of the probability, in Physics and Information

Octavio Obregón

Departamento de Física, Universidad de Guanajuato.

2022

The Entropy and Superstatistics

Boltzman-Gibbs (BG) statistics works perfectly well for classical systems with short range forces and relatively simple dynamics in equilibrium.

SUPERSTATISTICS: Beck and Cohen considered nonequilibrium systems with a long term stationary state that possesses a spatio-temporally fluctuating intensive quantity (temperature, chemical potential, energy dissipation). More general statistics were formulated.

The macroscopic system is made of many smaller cells that are temporarily in local equilibrium, β is constant. Each cell is large enough to obey statistical mechanics.

But has a different β assigned to it, according to a general distribution $f(\beta)$, from it one can get on effective Boltzmann factor

$$B(E) = \int_0^\infty d\beta \; f(\beta) e^{-\beta E}, \tag{1}$$

where E is the energy of a microstate associated with each of the considered cells. The ordinary Boltzmann factor is recovered for

$$f(\beta) = \delta(\beta - \beta_0). \tag{2}$$

One can, however, consider other distributions. Assume a Γ (or χ^2), distribution depending on a parameter p_l, to be identified with the probability associated with the macroscopic configuration of the system.

$$f_{p_l}(\beta) = \frac{1}{\beta_0 p_l \Gamma\left(\frac{1}{p_l}\right)} \left(\frac{\beta}{\beta_0}\frac{1}{p_l}\right)^{\frac{1-p_l}{p_l}} e^{-\beta/\beta_0 p_l}, \tag{3}$$

integrating over β

$$B_{p_l}(E) = (1 + p_l \beta_0 E)^{-\frac{1}{p_l}} . \tag{4}$$

By defining $S = k \sum_{l=1}^{\Omega} s(p_l)$ where p_l at this moment is an arbitrary parameter, it was shown that it is possible to express $s(x)$ by

$$s(x) = \int_0^\infty dy \frac{\alpha + E(y)}{1 - E(y)/E^*}, \tag{5}$$

where $E(y)$ is to be identified with the inverse function $\frac{B_{p_l(E)}}{\int_0^\infty dE' B_{p_l}(E')}$.

One selects $f(\beta) \to B(E) \to E(y), S(x)$ is then calculated.

For our distribution $\Gamma(\chi^2)$,

$$S = k \sum_{l=1}^{\Omega} \left(1 - p_l^{p_l}\right). \tag{6}$$

Its expansion gives

$$-\frac{S}{k} = \sum_{l=1}^{\Omega} \left[p_l \ln p_l + \frac{(p_l \ln p_l)^2}{2!} + \frac{(p_l \ln p_l)^3}{3!} + \cdots \right], \tag{7}$$

where the first term corresponds to the usual Boltzmann-Gibbs (Shannon) entropy.

Given the functional

$$\Phi = \frac{S}{k} - \gamma \sum_{l=1}^{\Omega} p_l - \beta \sum_{l=1}^{\Omega} p_l^{p_l+1} E_l, \tag{8}$$

maximizing Φ, p_l is obtained for S_I as

$$1 + \ln p_l + \beta E_l \left(1 + p_l + p_l \ln p_l\right) = p_l^{-p_l}. \tag{9}$$

And in a similar way for S_{II} as

$$1 + \ln p_l + \beta E_l \left(1 - p_l - p_l \ln p_l\right) = p_l^{p_l}. \tag{10}$$

The dominant term in these expressions correspond to the Boltzmann-Gibbs prediction, $p_l = e^{-\beta_0 E_l}$. In general, however, we cannot analytically express p_l as function of βE_l.

Assume now the equipartition condition $p_l = \frac{1}{\Omega}$, remember

$$-\frac{S}{k} = \sum_{l=1}^{\Omega} p_l \ln p_l \longrightarrow \frac{S_B}{k} = \ln \Omega. \tag{11}$$

In our case

$$S = k\Omega \left[1 - \frac{1}{\Omega^{\frac{1}{\Omega}}} \right], \tag{12}$$

in terms of S_B (the Boltzmann entropy), S reads

$$\frac{S}{k} = \frac{S_B}{k} - \frac{1}{2!} e^{-S_B/k} \left(\frac{S_B}{k} \right)^2 + \frac{1}{3!} e^{-2S_B/k} \left(\frac{S_B}{k} \right)^3 \cdots . \tag{13}$$

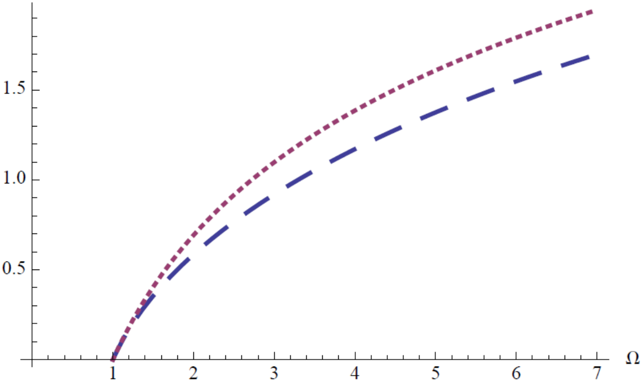


Figure: Entropies as function of (small) Ω. Blue dashed and red dotted lines correspond to $\frac{S}{k}$, and $\frac{S_B}{k}$, respectively ($p_l = 1/\Omega$ equipartition).

$$S_{I,II} = \sum_{n=1}^{\infty} (\mp)^{n+1} \frac{\ln^n \Omega}{n!\Omega^{n-1}} \tag{14}$$

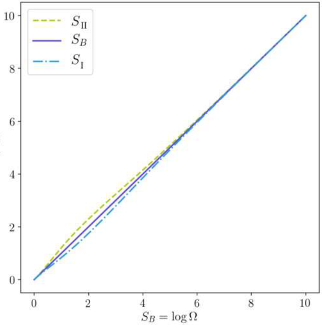


Figure: Entropies $S_{I,II}$ as a functions of S_B in a microcanonical ensemble.

$$S_q = \sum_{n=1}^{\infty} (-1)^n \frac{(q-1)^n}{(n+1)!} S_B^{n+1}. \tag{15}$$

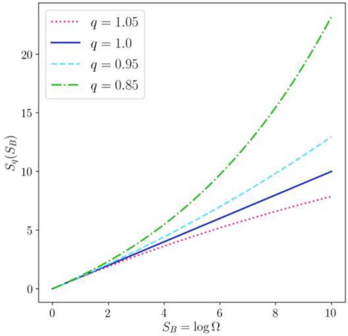


Figure: Tsallis entropy, $S_q(S_B)$, maximized for a microcanonical ensemble.

For the $f_{p_l}(\beta)$, $\Gamma(\chi^2)$ distribution we have shown that the Boltzmann factor can be expanded for small $p_l\beta_0 E$, to get

$$B_{p_l}(E) = e^{-\beta_0 E}\left[1 + \frac{1}{2}p_l\beta_0^2E^2 - \frac{1}{3}p_l^2\beta_0^3E^3\cdots\right]. \quad (16)$$

The Generalized Replica Trick

The corresponding generalized entanglement entropy to the entropy (6) is given by

$$\frac{S}{k} = Tr\left(I - \rho^{\rho}\right), \tag{17}$$

with ρ the density matrix, but this exactly corresponds to a "natural" generalization of the Replica trick namely

$$-\frac{S}{k} = \sum_{k \geq 1} \frac{1}{k!} \lim_{n \to k} \frac{\partial^k}{\partial n^k} Tr\rho^n. \tag{18}$$

According to Ted Jacobson's (and also E. Verlinde) proposal, we can reobtain gravitation from the entropy, for a modified entropy

$$S = \frac{A}{4 l_p^2} + \mathbf{s} \tag{19}$$

one gets a modified Newton's law

$$F = -\frac{GMm}{R^2} \left[1 + 4 l_p^2 \frac{\partial \mathbf{s}}{\partial A} \right]_{A = 4\pi R^2} \tag{20}$$

Coming back to our entropy and identifying $S_B = \frac{A}{4l_p^2}$ we get

$$F = -\frac{GMm}{R^2} + \frac{GMm\pi}{l_p^2}\left[1 - \frac{\pi R^2}{2l_p^2}\right] e^{-\frac{\pi R^2}{l_p^2}}. \tag{21}$$

Generalized gravitation? From Clausius relation (Jacobson)

$$\frac{\delta Q}{T} = \frac{2\pi}{\hbar}\int T_{ab}k^a k^b (-\lambda)\, d\lambda d^2 A, \tag{22}$$

and

$$\delta S_B = \eta \int R_{ab}k^a k^b (-\lambda)\, d\lambda d^2 A, \tag{23}$$

one gets Einstein's Equations. In our case, approximately one gets

$$\delta S = \delta S_B (1 - S_B). \tag{24}$$

A nonlocal gravity?

Quantum Superstatistics

The Bose-Einstein (BE), Fermi-Dirac (FD) and Classical (Boltzmann) distributions are given by $n_{BE} = \frac{1}{e^{(E_j-\mu)/kT}-1}$, $n_{FD} = \frac{1}{e^{(E_j-\mu)/kT}+1}$ and $n_{CL} = e^{-(E_j-\mu)/kT}$ respectively.

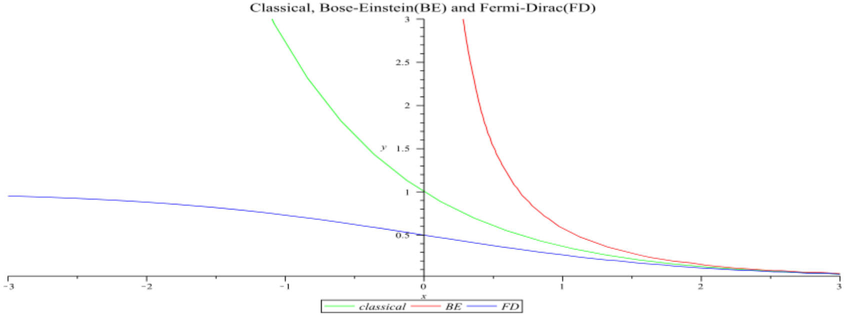


Figure: Comparison of Classic, BE and FD distributions.

In Fig. 5, p_l is drawn as a function of the reduced energy βE_l. We notice that for relative large values of βE_l the usual values for p_l coincide with the ones given by Eq. (9) and Eq. (10). As expected, they coincide also for $p_l \sim 1$.

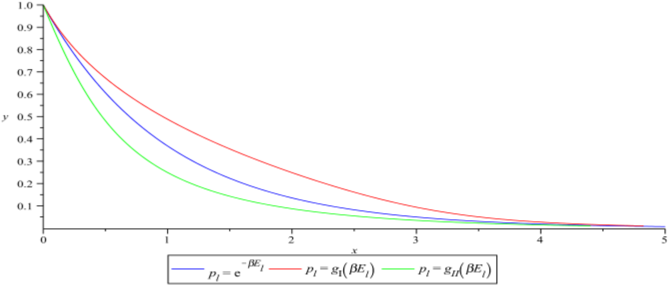


Figure: Comparison of the three probabilities. The blue line corresponds to the standard one $p_l = e^{-\beta E_l}$, red line to $p_l = g_I(\beta E_l)$ Eq. (9), and green line $p_l = g_{II}(\beta E_l)$, Eq. (10).

Now we will discuss the extension to the nonextensive statistical mechanics whose entropies depend only on the probability; corresponding to Bose-Einstein (BEO) and Fermi-Dirac (FDO) distributions. These entropies are: $S_I = k\sum_{l=1}^{\Omega}\left(1 - p_l^{p_l}\right)$, $S_I = k\sum_{l=1}^{\Omega}\left(p_l^{-p_l} - 1\right)$. As stated to them correspond

$$n_{I,II} = \frac{1}{g_{I,II}^{-1}\left(\alpha + \beta E_l\right) \pm 1}. \quad (25)$$

We will take $\beta E_l(p_l)$ and invert it and get the values of $p_{l_{I,II}} \equiv g_{I,II}(\alpha + \beta E_l)$. In this manner we will be able to calculate the occupancy number $n_{I,II}$ corresponding to the two entropies and their associated BEO and FDO.

It is shown that in both cases the BEO statistics differs from the usual BE statistics only slightly Fig. 6 and Fig. 7.

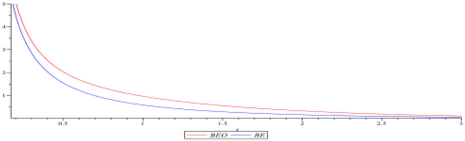


Figure: Comparison of BE y BEO (n_I). The red line corresponds to BEO usual , blue line to BE.

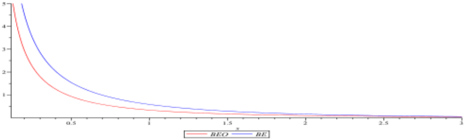


Figure: Comparison of BE y BEO (n_{II}). The red line corresponds to BEO usual , blue line to BE.

In the same way we can see that the FDO statistics differs from the FD statistics in a small amount, Fig. 8 and Fig. 9.

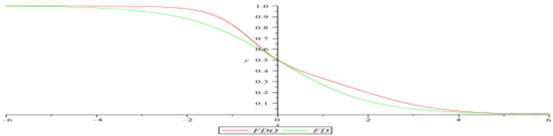


Figure: Comparison of FD with FDO (n_I). The red line corresponds to FDO, green line to FD.

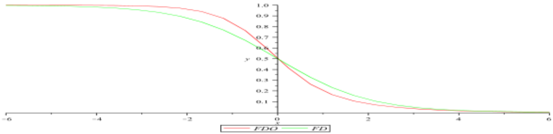


Figure: Comparison of FD with FDO (n_{II}). The red line corresponds to FDO, green line to FD.

The work of [Baez,Pollard] formulates the concept of Quantropy, deepening the analogy between Quantum Mechanics (Q.M.) and Statistical Mechanics (S.M.).

A path with probability in $a(x) = e^{-\frac{A(x)}{i\hbar}}$ Q.M. is mapped to a state with $p_l = e^{-\frac{E_l}{T}}$ probability in S.M.

Energy in S.M. is mapped to the action in Q.M., and the temperature to the Planck constant: $E \to A$, $T \to i\hbar$

Statistical fluctuations: T. Quantum fluctuations: $\hbar$.

The functional that upon finding its extrema leads to $a(x)$ is given by:

$$\Phi_{BP} = -\int_X a(x) \ln a(x) dx - \mu \int_X a(x) dx - \lambda \int_X a(x) S(x) dx. \tag{26}$$

Integrated quantropy

Let us discuss an integrated version of the Quantropy. First for the BG entropy, then for $S_{+(I)}$, $S_{-(II)}$ and S_q.
The change in distribution probabilities which arise from modified entropies in S.M. maps in Q.M. to modifications of the propagator.
The propagator between points in space-time (x_a, t_a) and (x_b, t_b) in Q.M. determine the probability amplitude of particles to travel from certain position to another position in a given time.
Modified entropies in S.M. lead to modified probability distributions, distinct probability amplitudes of propagation over all paths from (x_a, t_a) to (x_b, t_b) will arise from a modified Quantropy.

Integrated quantropy

As we write functionals to maximize and subject to constraints and obtain probability distributions, we can write the corresponding Quantropies to get the propagators K_+ and K_- and correspondingly the wave functions Ψ_+ and Ψ_-.

The standard BG statistics gives rise to the Quantropy:

$$Q_0 = -\int K(x) \ln K(x) dx$$

whose constrained extrema are obtained from:

$$\Phi_0 = -\int K(x) \ln K(x) dx + \alpha \int K(x) dx + \lambda \int S_{cl}(x) K(x) dx. \tag{27}$$

The extrema condition gives $K(x) = e^{-1-\alpha-\lambda S_{cl}(x)}$. For S_+ statistics the extrema are obtained from:

$$\Phi_+ = \int \left(1 - K(x)^{K(x)}\right) dx + \alpha \int K(x) dx + \lambda \int S_{cl}(x) K(x)^{K(x)+1} dx. \tag{28}$$

Integrated quantropy

The extrema condition gives rise to the modified propagator:

$$K_+(x) = N_+ e^{-\lambda S_{cl}} \left(1 - e^{-\lambda S_{cl}} (\lambda S_{cl})^2\right) + e^{-2\lambda S_{cl}} (\lambda S_{cl})^2 \left(-1 - 2(\lambda S_{cl}) + 3(\lambda S_{cl})^2\right) + \cdots \quad (29)$$

We consider also S_- statistics and Tsallis S_q statistics, the same sketched procedure gives rise to the set of modified propagators:

$$K_-(x) = N_- e^{-\lambda S_{cl}} \left(1 + e^{-\lambda S_{cl}} (\lambda S_{cl})^2\right) - e^{-2\lambda S_{cl}} (\lambda S_{cl})^2 \left(-1 - 2(\lambda S_{cl}) + 3(\lambda S_{cl})^2\right) + \cdots \quad (30)$$

$$K_q(x) = N_q \exp_q(-\lambda S_{cl}(x)) = N_q (1 - (1-q)\lambda S_{cl})^{\frac{1}{1-q}} . \quad (31)$$

Free particle

We write a modified propagator up to third order for the free particle in the case of the statistics S_+, S_- and S_q for $q < 1$ and $q > 1$. The values of q are considered in order to compare the different propagators.
The classical action is given by:

$$S_{cl} = \frac{mx^2}{2t}. \tag{32}$$

The standard normalized free particle propagator is given by:

$$K_0(x) = \sqrt{\frac{m}{2\pi\hbar t}} \exp\left(\frac{imx^2}{2\hbar t}\right). \tag{33}$$

This is obtained from the extrema of the BG Quantropy.

Free particle

Consider now S_+ statistics Quantropy extrema to obtain:

$$K_+(x) = N_+ \exp\left(\frac{imx^2}{2\hbar t}\right)\left(1 + \exp\left(\frac{imx^2}{2\hbar t}\right)\left(\frac{imx^2}{2\hbar t}\right)^2 + \cdots\right) \tag{34}$$

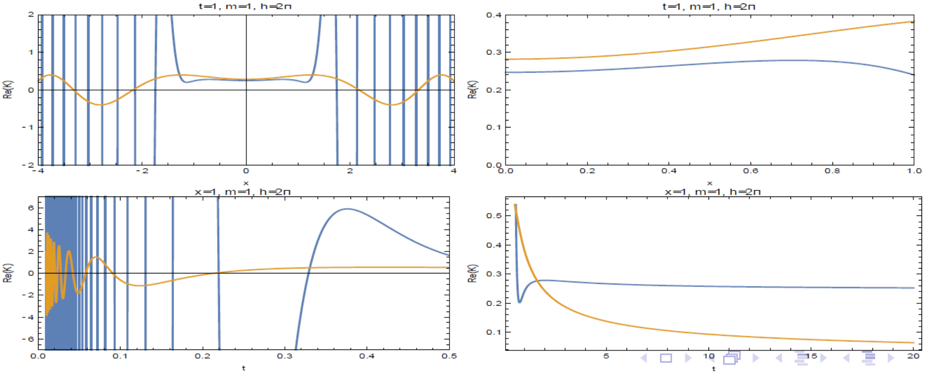
Similarly for S_- and S_q statistics Quantropies extrema we get:

$$K_-(x) = N_- \exp\left(\frac{imx^2}{2\hbar t}\right)\left(1 - \exp\left(\frac{imx^2}{2\hbar t}\right)\left(\frac{imx^2}{2\hbar t}\right)^2 + \cdots\right) \tag{35}$$

$$K_q(x) = N_q\left(1 + (q-1)\left(\frac{imx^2}{2\hbar t}\right)\right)^{\frac{1}{1-q}}. \tag{36}$$

Free-Particle. S_+ Quantropy

Real parts of the modified propagator (blue line) vs. standard propagator (yellow line), for the free particle for the modified statistics S_+. The quantum regime is given by $S_{cl} \sim \hbar$ and it translates for fixed $x = 1$ in $t \geq 1/2$ for fixed $t = 1$ translates in $x^2 \leq 2$. We set $\hbar = m = 1$.



Generalized Noiseless Coding Theorems

Shannon seminal paper (1948); a firm setting of the foundations of Classical Information Theory. Suppose we have n input symbols $x_1, x_2, ..., x_n$ with probabilities $p_1, p_2, ..., p_n$ which we wish to encode. Suppose further that there is an alphabet of D symbols into which the input symbols are to be encoded. Let x_i be represented by a sequence of l_i characters from the alphabet. It was shown that there is a uniquely decipherable code with lengths $l_1, l_2, ..., l_n$ if and only if

$$C = D \sum_i D^{-l_i} \leq 1. \tag{37}$$

There are, however many lengths that satisfy (37), called the Kraft inequality.

For the standard Shannon entropy one optimises

$$J = L + \gamma C = \sum_i p_i l_i + \gamma \sum_i D^{-l_i}, \qquad \gamma \in \Re, \tag{38}$$

where γ is a Lagrange multiplier to be selected adequately. One gets $l_i^* = \log_D p_i$, this *is the optimal length*, it correspond then to the entropy.

In 1965 L. Campbell generalized C and even L to obtain on appropriate coding theorem for the Rényi entropy. Then following a similar approach as that of Campbell, in the last two decades several papers appeared that generalized this coding theorem for q-statistics, (for Havrda-Charvát-Tsallis). We show here a proposal to find, generalized coding theorems for $S_{I,II}$ which we rename as $H_D^{(\pm)}(X)$.
Following our approach one can also find appropriate demonstrations for the coding theorems corresponding to the Rényi entropy and the q-statistics entropy. These are not shown here.

We proceed analogously as in Shannon, for $H_D^{(\pm)}(X)$. The problem consists of choosing a code that minimises

$$C^{(\pm)} = \sum_i \sum_j a_j^{(\pm)} \Gamma\left[j+1, \log D^{-l_i}\right] \leq \text{constant}, \tag{39}$$

where the $a_j^{(\pm)}$ are real coefficients corresponding to generalized exponentials

$$\exp^{(\pm)}(x) \equiv \exp(-x) \sum_{j=0}^{\infty} a_j^{(\pm)} x^j, \qquad a_j^{(\pm)} \in \Re \tag{40}$$

$\Gamma(.,.)$ is the incomplete gamma function.

Solving such optimization problem consist of choosing the mean length defined by

$$L^{(\pm)} = \sum_i p_i^{(\pm)} l_i, \tag{41}$$

the functional to be minimized,

$$J^{(\pm)} = L^{(\pm)} + \gamma C^{(\pm)}, \qquad \gamma \in \Re. \tag{42}$$

We draw attention to remark that the standard theory is straightforwardly recoverable from our scheme in the limit $\Omega \to \infty$, which means that the Shannon quantity p_i can be obtainable from the transition $p_i = \lim_{\Omega\to\infty} p_i^{(\pm)}$, therefore leading to the classical functional (38) as

$$\lim_{\Omega\to\infty} J^{(\pm)} = J. \tag{43}$$

To determine the optimal partial lengths $l_i^{(\pm)^*}$, we now differentiate (42) with respect to l_i and equate to zero to obtain

$$\frac{\partial J^{(\pm)}}{\partial l_i} = \frac{\partial L^{(\pm)}}{\partial l_i} + \gamma \frac{\partial C^{(\pm)}}{\partial l_i} = 0, \tag{44}$$

the equality is satisfied for all i iff

$$l_i^{(\pm)^*} = -\log_D^{(\pm)}\left(p_i^{(\pm)}\right). \tag{45}$$

The lengths $L^{(\pm)}$ should be lower and upper bounded by the entropy measures themselves. Thus we are entitled to introduce the following theorems.

Theorem (1)

The expected lengths defined by Eq. (41) for a D-ary alphabet regarding the entropies $H_D^{(\pm)}(X)$, satisfy $L^{(\pm)} \geq H_D^{(\pm)}(X)$, with equality iff $l_i^{(\pm)^*} = -\log_D^{(\pm)}\left(p_i^{(\pm)}\right)$ for every i.

Theorem (2)

For a D-ary alphabet and a source distribution X let $l_i^{(\pm)^*}$ be the optimal partial lengths that solve the optimisation problem (42), where the associated mean lengths are defined by Eq. (41). Then

$$H_D^{(\pm)}(X) \leq L^{(\pm)} \leq H_D^{(\pm)}(X) + 1. \quad (46)$$

Certainly, the expected lengths $L^{(\pm)}$ are such that satisfy $H_D^{(\pm)}(X) \leq L^{(\pm)} \leq H_D^{(\pm)}(X) + 1$. Nonetheless the optimal code in view of $H_D^{(\pm)}$ can only be better that the prescribed by $L^{(\pm)}$, thus one is addressed to the Theorem 2. As an example, we have generated a random process with uncorrelated sources, see Figure 10, to serve as datasets to compute the expected lengths $L^{(\pm)}$ as well as the usual length, L, in classical information theory. For the process in Fig. 4, as for Shannon entropy, $H_{D=2}$, the expected length is $L^* = 6.64$ bits, whereas the expected lengths as for entropies $H_{D=2}^{(+)}$ and $H_{D=2}^{(-)}$ the expected lengths are $L^{(+)*} = 5.91$ bits and $L^{(-)*} = 6.39$ bits, in that order. That means that a more efficient transmission process would result from entropy $H_{D=2}^{(+)}$ in comparison to a code computed through $H_{D=2}$. However as the number of random events increases, the lengths $L^{(\pm)*}$ tend to coincide with L^*.

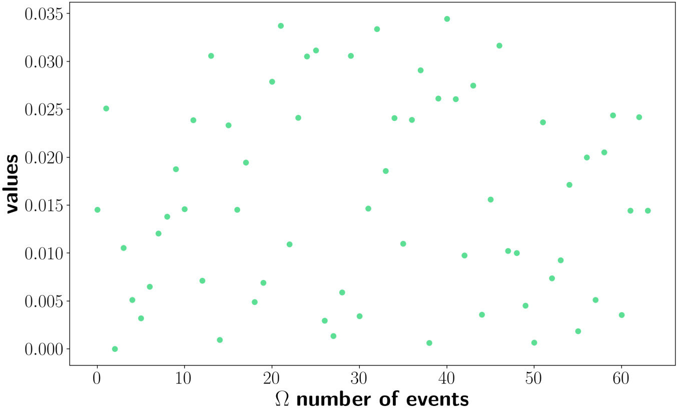


Figure: Random processes for $\Omega = 2^6$ events.

Capacity

The amount of information that can be obtained about x observing y. It is used to maximize the amount of information showed between sent and received signals. The tight upper bound on the role at which information can be reliably transmitted over a communication channel.

The capacity of a communication channel as obtained by Shannon:

Let $X = \{x_1, x_2, ..., x_n\}$ be an input to the channel with probabilities $p(x_1)$, $p(x_2)$,..., $p(x_n)$ and let $Y = \{y_1, y_2, ..., y_n\}$ be its output. The channel itself is entirely described by a set of conditional probabilities $\{p(y_j|x_i)\}$, of receiving y_j if x_i was sent. The capacity is defined to be the maximum over all the probability distributions of $p(x_1)$, $p(x_2)$,..., $p(x_n)$ as the Shannon mutual information between the input X and output Y , $I(X; Y)$

$$C = \max_{p(x_i)} I(X:Y) = \max_{p(x_i)} \{S(Y) - S(Y|X)\}, \tag{47}$$

where $S(Y|X)$ is the conditional entropy

$$S(Y|X) = \sum p(x_i) S(Y|X = x_i) = \sum p(x_i) \left(\sum p(y_j|x_i) \log p(y_j|x_i) \right) \tag{48}$$

It tell us how much the uncertainty in the input $S(X)$ is reduced by measuring the output $S(X|Y)$ and the maximum of this information (of the uncertainty reduction) is called the capacity C.

The previous equations (47), (48) tell us how to compute channel capacities for other entropies. This has been done in several works for the Rényi entropy and Tsallis entropy. In particular for the binary symmetric channel.

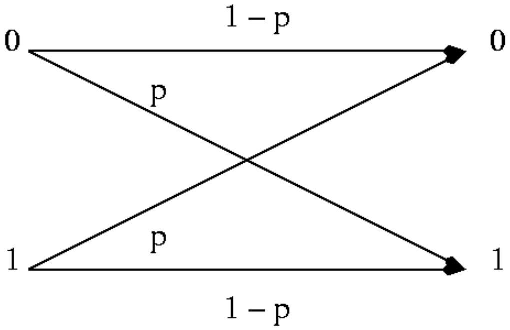
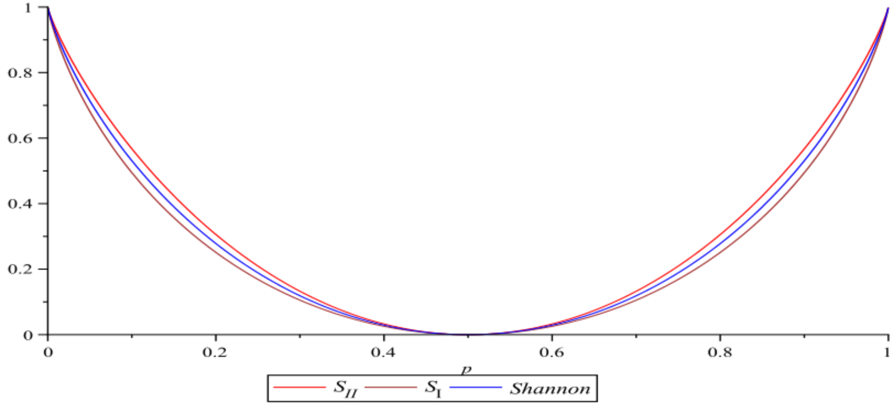
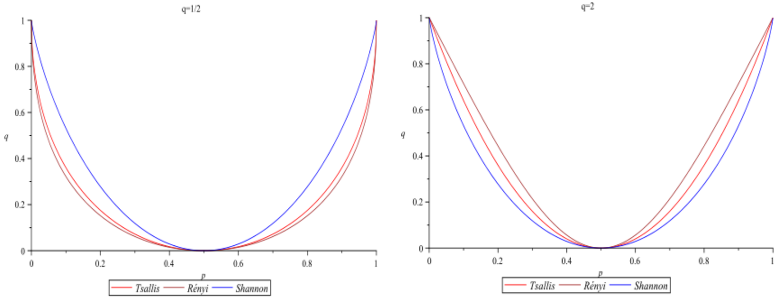


Figure: Binary Symmetric Channel.

In most of the previous works, one can not compare the results with the Shannon channel capacity for this same case. In some of them it has correctly demanded that this capacities should be zero for $p = 1/2$, one most however further demand that $C = 1$ for $p = 0$ and $p = 1$.
Under these necessary conditions, we have correctly calculated the channel capacities for the symmetric channel. We do not present the associate expression but only the resulting graphs. As can be observed, S_{II} gives a slight largest capacity than Shannon.

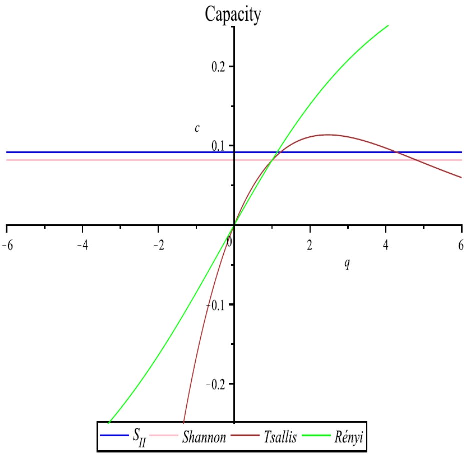

1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
p
S_{II}
S_I
Shannon



Figure

In Figs. 12 one notice that the q-dependent Rényi and Tsallis entropies give for $q = 2$ better channel capacities than Shannon and for $q = 1/2$ Shannon channel capacity is larger.

Fig. 13 is for a fixed probability $p = 1/3$ and is a function of the parameter q.



Figure

For this reason Shannon and S_{II} (S_I is not draw) have unique stable values. A relative small change in q can drastically change the value at the channel capacity. This could probably point out to an old (1981) argument by Berhard Lesche, in connection with the Rényi entropy, in which he claims that a necessary condition for a quantity $C(q)$ to be observable is that its values do not change dramatically if the state of the system in consideration is changed by an unobservable small amount parameterized by $q \to q + \delta q$.

Uncertainty principle from entropy

- The Heisenberg uncertainty principle, is a consequence of the entropic uncertainty principle.
- The uncertainty principle needs to be extended to a Generalized Uncertainty Principle (GUP) to describe the quantum effects of gravity. i.e. string theory, a Gedankenexperiment of black-hole difraction.
- Can a GUP be deduced from the entropic uncertainty when generalized entropies are considered?
- We show that GUP is a consequence of the entropic principle for the statistics of generalized entropies depending only on the probability.
- The corrections to the energy levels for the simple harmonic oscillator with this method match the usual ones founded in GUP.
- The corrections show a universality with respect to several entropy measures.

References I

[1] Nana Cabo Bizet, Octavio Obregón, and Wilfredo Yupanqui.
Modified entropies as the origin of generalized uncertainty principles.
arXiv preprint arXiv:2206.14166, 2022.

[2] Nana Cabo Bizet, César Damián, Octavio Obregón, and Roberto Santos-Silva.
Quantum implications of non-extensive statistics.
Frontiers in Physics, page 62, 2021.

[3] Nana Cabo Bizet, Jesus Fuentes, and Octavio Obregon.
Generalised asymptotic equivalence for extensive and non-extensive entropies.
arXiv e-prints, page arXiv:1904.07858, April 2019.

[4] Nana Cabo Bizet and Octavio Obregón.
Exploring the gauge/gravity duality of a generalized von neumann entropy.
The European Physical Journal Plus, 133(2):1–10, 2018.

[5] Jesús Fuentes, José Luis López, and Octavio Obregón.
Generalized fokker-planck equations derived from nonextensive entropies asymptotically equivalent to boltzmann-gibbs.
Phys. Rev. E, 102:012118, Jul 2020.

[6] Jesús Fuentes and Octavio Obregón.
On how generalised entropies without parameters impact information optimisation processes.
arXiv preprint arXiv:2103.11759, 2021.

References III

[7] A. Gil-Villegas, O. Obregón, and J. Torres-Arenas.
Computer simulation of effective potentials for generalized boltzmann-gibbs statistics.
Journal of Molecular Liquids, 248:364–369, 2017.

[8] JL López-Picón, O Obregón, and J Ríos-Padilla.
Ensemble fluctuations related to generalized (probability-dependent) entropies and their work fluctuation theorem.
The European Physical Journal Plus, 137(3):1–11, 2022.

[9] J.L. López, O. Obregón, and J. Torres-Arenas.
Thermodynamic geometry for a non-extensive ideal gas.
Physics Letters A, 382(17):1133–1139, 2018.

References IV

[10] Aldo Martínez-Merino, Octavio Obregón, and Michael P. Ryan. Modified entropies, their corresponding newtonian forces, potentials, and temperatures. *Phys. Rev. D*, 95:124031, Jun 2017.

[11] Aldo Martínez-Merino, Octavio Obregón, and Michael P. Ryan. Newtonian black holes: Particle production, "hawking" temperature, entropies and entropy field equations, 2016.

[12] O. Obregón. Generalized information and entanglement entropy, gravitation and holography. *International Journal of Modern Physics A*, 30(16):1530039, 2015.

[13] Octavio Obregón and Alejandro Gil-Villegas.
Generalized information entropies depending only on the probability distribution.
Phys. Rev. E, 88:062146, Dec 2013.

[14] Octavio Obregón, J Torres-Arenas, and A Gil-Villegas.
H-theorem and thermodynamics for generalized entropies that depend only on the probability.
arXiv preprint arXiv:1610.06596, 2016.

[15] Octavio Obregón.
Superstatistics and gravitation.
Entropy, 12(9):2067–2076, 2010.

References VI

[16] Octavio Obregón, José Luis López, and Marco Ortega-Cruz.
On quantum superstatistics and the critical behavior of nonextensive ideal bose gases.
Entropy, 20(10), 2018.