Compact stars in $f(R)$ modified gravity

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Brief introduction

General Relativity (GR)

$$S = \frac{1}{16\pi}\int \sqrt{-g}R d^4x + S_m, \tag{1}$$

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu} \qquad \Rightarrow \qquad R = -8\pi T. \tag{2}$$

"*Matter tells space how to curve, and curved space tells matter how to move*"

Why modify the gravitational action of GR?

- Early attempts: Weyl (1919) and Eddington (1923).
- Higher-order actions are renormalizable: Utiyama (1962), Stelle (1977).
- It is feasible to obtain accelerated expansion of the Universe in modified gravity without the need to introduce fluids with exotic properties such as dark energy.

Stellar structure equations in GR

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -e^{2\psi}dt^2 + e^{2\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (3)$$

$$T_{\mu\nu} = (\rho + p_t)u_\mu u_\nu + p_t g_{\mu\nu} - \sigma k_\mu k_\nu, \quad (4)$$

$$u_\mu u^\mu = -1, \qquad k_\mu k^\mu = 1, \qquad u_\mu k^\mu = 0, \quad (5)$$

where ρ is the energy density, p_r the radial pressure, p_t the tangential pressure and $\sigma \equiv p_t - p_r$ is the anisotropy factor. Accordingly, we can write $u^\mu = e^{-\psi}\delta_0^\mu$, $k^\mu = e^{-\lambda}\delta_1^\mu$ and the trace of the energy-momentum tensor (4) takes the form $T = -\rho + 3p_r + 2\sigma$.

Several sources of anisotropy:

- Strong magnetic fields
- Solid or superfluid cores
- Crystallization of the core
- Pion condensation, etc.

See Refs. [Phys. Rep. **286** 53 (1997), arXiv:2008.05331 [gr-qc]]

TOV equations

$$\frac{dm}{dr} = 4\pi r^2 \rho, \tag{6}$$

$$\frac{dp_r}{dr} = -(\rho + p_r)\left(\frac{m}{r^2} + 4\pi r p_r\right)\left(1 - \frac{2m}{r}\right)^{-1} + \frac{2}{r}\sigma, \tag{7}$$

$$\frac{d\psi}{dr} = -\frac{1}{\rho + p_r}\frac{dp_r}{dr} + \frac{2\sigma}{r(\rho + p_r)}, \tag{8}$$

with boundary conditions:

$$m(0) = 0, \qquad \rho(0) = \rho_c, \qquad \psi(r_{\text{sur}}) = \frac{1}{2}\ln\left[1 - \frac{2M}{r_{\text{sur}}}\right], \tag{9}$$

for a given EoS $p_r = p_r(\rho)$ and a defined anisotropy relation for σ. The metric function $\lambda(r)$ is obtained by means of the relation

$$e^{-2\lambda} = 1 - \frac{2m}{r}. \tag{10}$$

Surface: $p_r(r = R) = 0$, and $M \equiv m(R)$ is the total gravitational mass.

Field equations

$$S = \frac{1}{16\pi} \int \sqrt{-g} f(R) d^4x + S_m, \tag{11}$$

by varying action (11) with respect to the metric:

$$\boxed{f_R R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f - \nabla_\mu \nabla_\nu f_R + g_{\mu\nu} \Box f_R = 8\pi T_{\mu\nu}}, \tag{12}$$

where $f_R(R) \equiv df(R)/dR$, and taking the trace of (13)

$$3\Box f_R(R) + R f_R(R) - 2f(R) = 8\pi T, \tag{13}$$

where $\Box \equiv \nabla_\mu \nabla^\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left[\sqrt{-g} \partial^\mu\right]$ is the d'Alembertian operator.

$T = 0$ no longer implies that $R = 0$ as in pure GR

Stellar structure in $f(R)$ gravity

For the line element (3) and energy-momentum tensor (4), the modified TOV equations in $f(R)$ gravity are given by [arXiv:2202.04467 [gr-qc]]

$$
\begin{aligned}
\frac{d\psi}{dr} &= \frac{1}{2r(2f_R + rR'f_{RR})}\left[r^2 e^{2\lambda}(16\pi p_r + f - Rf_R) + 2f_R\left(e^{2\lambda}-1\right) - 4rR'f_{RR}\right],\\
\frac{d\lambda}{dr} &= \frac{1}{2r(2f_R + rR'f_{RR})}\left\{2f_R\left(1-e^{2\lambda}\right) + \frac{r^2e^{2\lambda}}{3}\left[16\pi(2\rho + 3p_r + 2\sigma) + Rf_R + f\right]\right.\\
&\qquad \left. + \frac{rR'f_{RR}}{f_R}\left[2f_R\left(1-e^{2\lambda}\right) + \frac{r^2e^{2\lambda}}{3}\left(16\pi\rho + 16\pi\sigma + 2Rf_R - f\right) + 2rR'f_{RR}\right]\right\},\\
\frac{d^2R}{dr^2} &= \frac{1}{3f_{RR}}\left\{e^{2\lambda}\left[8\pi(-\rho + 3p_r + 2\sigma) + 2f - Rf_R\right] - 3R'^2 f_{RRR}\right\} + \left(\lambda' - \psi' - \frac{2}{r}\right)R',\\
\frac{dp_r}{dr} &= -(\rho + p_r)\psi' + \frac{2}{r}\sigma,
\end{aligned}
$$

where the prime denotes derivative with respect to r.

Inside the star

$$\rho(0) = \rho_c, \qquad \psi(0) = \psi_c, \qquad \lambda(0) = 0, \qquad R(0) = R_c, \qquad R'(0) = 0.$$

Outside the star both density and radial pressure vanish ($\rho = p_r = 0$). Now we have to use the junction conditions on the surface of the star:

$$\psi_{in}(r_{\text{sur}}) = \psi_{out}(r_{\text{sur}}), \qquad \lambda_{in}(r_{\text{sur}}) = \lambda_{out}(r_{\text{sur}}),$$
$$R_{in}(r_{\text{sur}}) = R_{out}(r_{\text{sur}}), \qquad R'_{in}(r_{\text{sur}}) = R'_{out}(r_{\text{sur}}).$$

On the other hand, we can define a mass function $m(r)$ characterizing the mass enclosed within the radius r according to the relation

$$e^{-2\lambda} = 1 - \frac{2m}{r}, \tag{14}$$

where

$$m = 4\pi \int \rho r^2 dr + \frac{1}{2} \int \left\{ \frac{(1 - f_R)}{r^2} \frac{d}{dr} \left[r(1 - e^{-2\lambda}) \right] + \frac{1}{e^{2\lambda}} \left[\left(\frac{2}{r} - \lambda' \right) R' f_{RR} + R'' f_{RR} + R'^2 f_{RRR} \right] + \frac{1}{2}(R f_R - f) \right\} r^2 dr. \tag{15}$$

Constraints on the scalar curvature and the mass parameter come from the asymptotic flatness requirement:

$$\lim_{r\to\infty} R(r) = 0, \qquad \lim_{r\to\infty} m(r) = \text{constant}. \tag{16}$$

Total gravitational mass

$$M \equiv \lim_{r\to\infty} \frac{r}{2}\left(1 - \frac{1}{e^{2\lambda}}\right). \tag{17}$$

$f(R) = R^{1+\epsilon}$ gravity

* Power-law models given by $f(R) \sim R^n$ (where $n \in \mathbb{R}$) are related to the existence of Noether symmetries [arXiv:gr-qc/0703067].
* It's a good candidate to solve both the dark energy problem at cosmological level and the dark matter one at galactic scale.
* We assume that $n = 1+\epsilon$ so that we can study small deviations with respect to GR for $|\epsilon| \ll 1$. Then, $R^{1+\epsilon} \simeq R + \epsilon R \ln R$.
* Such correction emerges in one-loop regularization and renormalization process in curved spacetime.

Equation of state (EoS) and anisotropy profile

To explore quark stars in $R^{1+\epsilon}$ gravity, we employ the MIT bag model EoS for the dense matter involved, given by

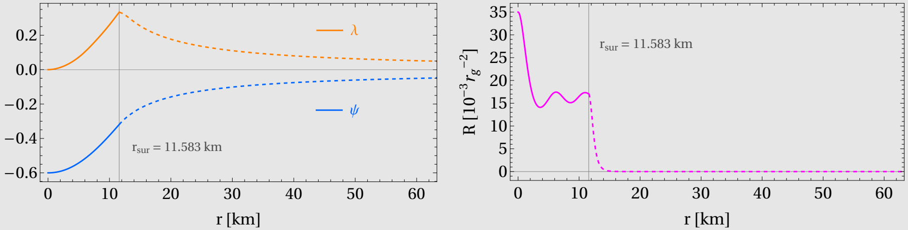
$$p_r = b(\rho - 4B). \tag{18}$$

It describes a self-gravitating fluid composed by up, down, and strange quarks. The constant b varies from 0.28 to 1/3, and the bag constant B lies in the range $0.982B_0 < B < 1.525B_0$ where $B_0 = 60\,\mathrm{MeV/fm^3}$. We will consider the particular case $b = 1/3$ and $B = B_0$.
To model anisotropic matter inside compact stars, we will use the anisotropy ansatz suggested by Horvat *et al.* [arXiv:1010.0878 [gr-qc]]:

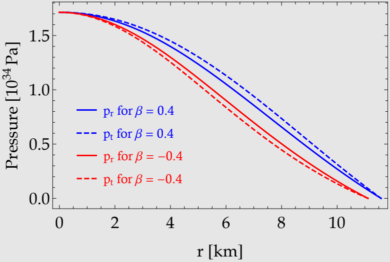
$$\sigma = \beta p_r \mu = \beta p_r \left(2m/r\right), \tag{19}$$

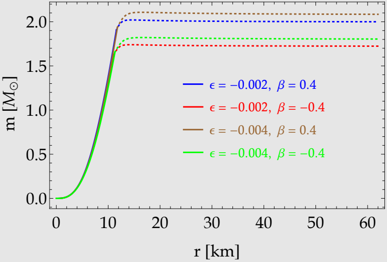
The stellar fluid becomes isotropic at the origin since $\mu \sim r^2$ when $r \to 0$. In the non-relativistic limit, the effect of anisotropy vanishes in the hydrostatic equilibrium equation.

Numerical solution of the set of modified TOV equations for a given central density $\rho_c = 1.0 \times 10^{18}\,\mathrm{kg/m^3}$ with MIT bag model EoS (18) and anisotropy profile (19) in $R^{1+\epsilon}$ gravity model, where we considered $\epsilon = -0.002$ and $\beta = 0.4$:



By setting $p_r(r) = 0$, we obtain the radius of the star $r_{\mathrm{sur}} = 11.583\,\mathrm{km}$. At the surface: $m_{\mathrm{sur}} = 1.906 M_\odot$ and at infinity: $M = 1.996 M_\odot$. The Ricci scalar goes to zero as we move away from the stellar surface according to the asymptotic flatness requirement.

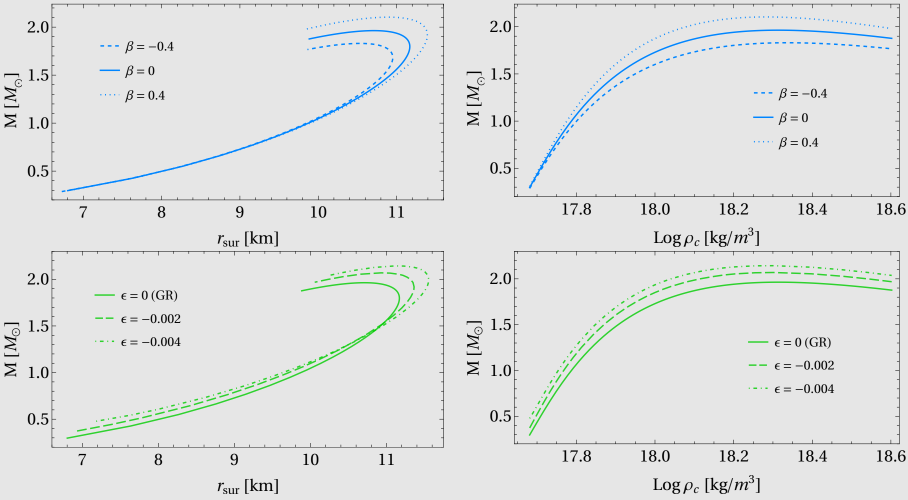




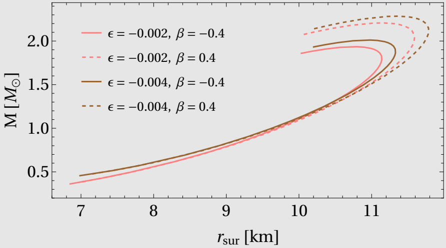
Stellar configurations with central energy density $\rho_c = 1.0 \times 10^{18}\,\mathrm{kg/m^3}$ for different values of ϵ and β:

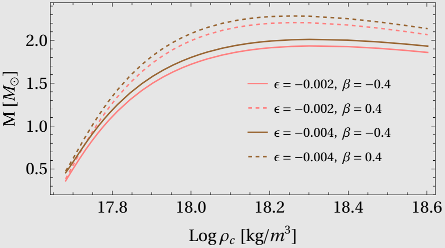
Parameters	$r_{\rm sur}$ [km]	$m_{\rm sur}$ [$M_\odot$]	M [$M_\odot$]
$\epsilon = 0$ (GR), $\beta = 0$	11.151	1.729	1.729
$\epsilon = -0.002$, $\beta = -0.4$	11.111	1.628	1.721
$\epsilon = -0.002$, $\beta = 0$	11.344	1.759	1.851
$\epsilon = -0.002$, $\beta = 0.4$	11.583	1.906	1.996
$\epsilon = -0.004$, $\beta = -0.4$	11.299	1.656	1.801
$\epsilon = -0.004$, $\beta = 0$	11.539	1.791	1.934
$\epsilon = -0.004$, $\beta = 0.4$	11.779	1.936	2.080

Mass-radius diagrams



The blue curves in the upper panels represent anisotropic solutions for $\epsilon = 0$. Green lines in the middle panels correspond to isotropic solutions ($\beta = 0$) for different values of ϵ.





- For β fixed and ϵ varying, significant deviations from GR appear in both high-mass and low-mass regions.
- On the other hand, for β varying and ϵ fixed, the total gravitational mass of quark stars undergoes very slight changes at low central densities and the most substantial changes occur at higher densities due to the anisotropic pressure.
- The "$\epsilon R \ln R$" term allows for an increase in the maximum-mass values as ϵ becomes more negative.
- Some combinations of ϵ and β are capable of generating maximum masses above $2M_\odot$.

Starobinsky model

- It has gained great interest in recent years because its inflationary predictions are in good agreement with the 2018 Planck measurements of the cosmic microwave background anisotropies [arXiv:1807.06211 [astro-ph.CO]].
- However, such a model does not provide a late-time accelerated expansion.
- For isotropic stars ($\sigma = 0$) and $f(R) = R + \alpha R^2$, the modified TOV equations become

$$\frac{d\psi}{dr} = \frac{1}{4r(1+2\alpha R+\alpha r R')}\left[r^2 e^{2\lambda}(16\pi p - \alpha R^2) + 2(1+2\alpha R)\left(e^{2\lambda}-1\right) - 8\alpha r R'\right],$$

$$\frac{d\lambda}{dr} = \frac{1}{4r(1+2\alpha R+\alpha r R')}\left\{2(1+2\alpha R)\left(1-e^{2\lambda}\right) + \frac{r^2 e^{2\lambda}}{3}\left[16\pi(2\rho+3p) + 2R + 3\alpha R^2\right]\right.$$
$$\left. + \frac{2\alpha r R'}{1+2\alpha R}\left[2(1+2\alpha R)\left(1-e^{2\lambda}\right) + \frac{r^2 e^{2\lambda}}{3}(16\pi\rho + R + 3\alpha R^2) + 4\alpha r R'\right]\right\},$$

$$\frac{d^2 R}{dr^2} = \frac{e^{2\lambda}}{6\alpha}\left[R + 8\pi(3p-\rho)\right] + \left(\lambda' - \psi' - \frac{2}{r}\right)R',$$

$$\frac{dp}{dr} = -(\rho+p)\psi',$$

Equation of state (EoS)

EoS for a system composed by up, down, and strange quarks (at zero temperature) was obtained within pQCD by Freedman and McLerran [Phys. Rev. D 16, 1169 (1977), Phys. Rev. D 17, 1109 (1978)].

Pocket formula

$$\boxed{p = p_{\rm SB}(\mu_B)\left(c_1 - \frac{a(X)}{(\mu_B/{\rm GeV}) - b(X)}\right)}, \tag{20}$$

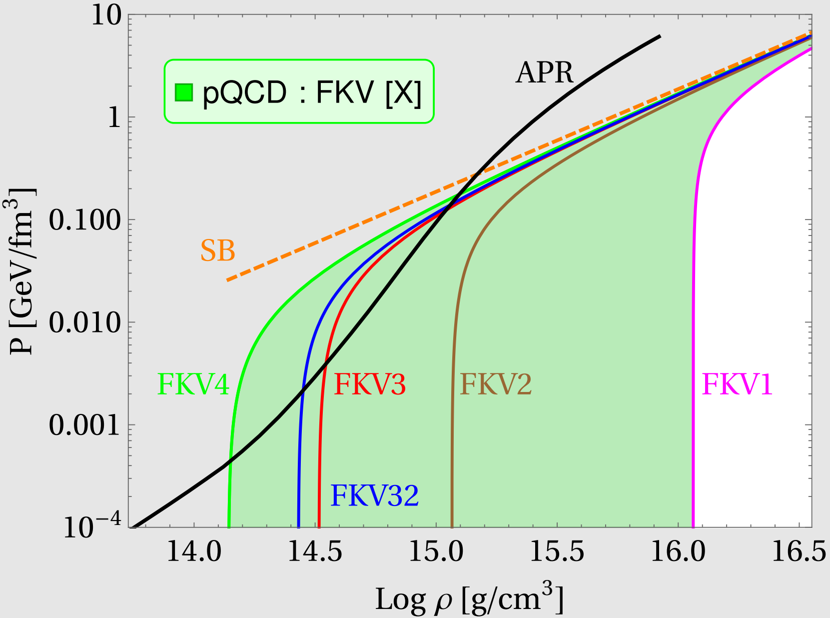
where X is the is the renormalization scale parameter and is usually taken to vary between 1 and 4 [E. S. Fraga, A. Kurkela, and A. Vuorinen, Astrophys. J. Lett. 781, L25 (2014)] $\Rightarrow$ FKV$[X]$

From Eq. (20), one can obtain a bag-like EoS

$$p = \frac{1}{3}\left(\rho - t^{\mu}_{\mu}(\mu_B, X)\right), \tag{21}$$

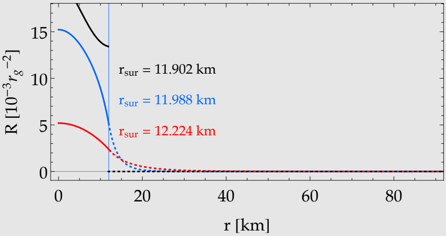
where ρ is the energy density and

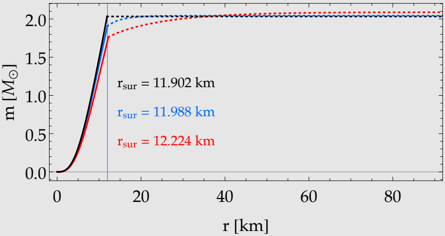
$$t^{\mu}_{\mu}(\mu_B, X) = \frac{\mu_B}{\rm GeV}\frac{a(X)}{\left[(\mu_B/{\rm GeV}) - b(X)\right]^2}. \tag{22}$$


pQCD : FKV [X]
APR
SB
FKV4
FKV3
FKV2
FKV1
FKV32
P [GeV/fm³]
Log ρ [g/cm³]
10
1
0.100
0.010
0.001
10⁻⁴
14.0
14.5
15.0
15.5
16.0
16.5

Numerical results [arXiv:2112.09950 [gr-qc]]

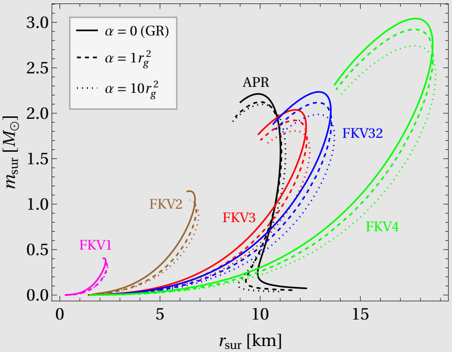
Ricci scalar and mass function for a quark star obtained from the FKV3 EoS with central density $\rho_c = 1.5 \times 10^{15}\text{g/cm}^3$ within the Starobinsky model with $\alpha = 1r_g^2$ and $\alpha = 10r_g^2$

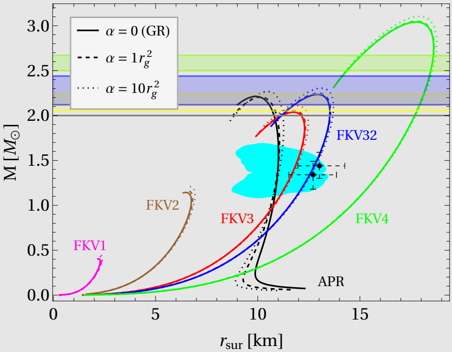




By setting $p(r) = 0$, we obtain the radius of the star r_{sur}.
At infinity, $M = 2.044M_\odot$ (for $\alpha = 1r_g^2$) and $M = 2.089M_\odot$ (for $\alpha = 10r_g^2$).
The **GR case** is shown by black lines, where the total mass is $M = 2.033M_\odot$.

Mass-radius relations





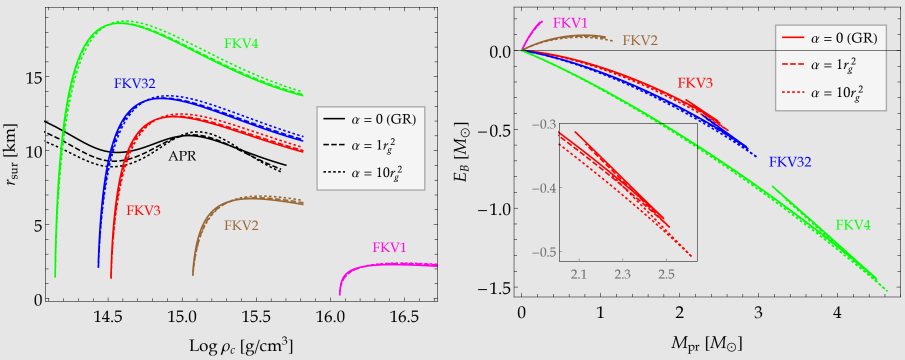
m_{sur} decreases significantly compared to GR for moderate values of α.

Observational measurements of the masses of the highly massive NS pulsars J0740+6620 and J2215+5135.

Lower mass of the compact object detected by the GW190814 event.

Mass-radius constraint from the GW170817 event.

Black dots: NICER measurements for PSR J0030+0451.



- ⋆ The impact of the αR^2 term on the hadronic star is larger in r_{sur}, although mostly pronounced at low central-mass densities.
- ⋆ Binding energy: $E_B = M - M_{\text{pr}}$, with M_{pr} being the proper mass

$$M_{\text{pr}} = 4\pi m_{\text{B}} \int_0^{r_{\text{sur}}} e^{\lambda(r)} r^2 n_B(r) dr, \tag{23}$$

where $n_B(r)$ is the baryon number density.

Charged quark stars in $f(R)$ gravity [arXiv:2206.03878 [gr-qc]]

$$S=\int d^4x\sqrt{-g}\left[\frac{1}{16\pi}f(R)+\mathcal{L}_m+\mathcal{L}_e\right], \tag{24}$$

$$f_R R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}f-\nabla_\mu\nabla_\nu f_R+g_{\mu\nu}\Box f_R=8\pi(\mathcal{M}_{\mu\nu}+\mathcal{E}_{\mu\nu}), \tag{25}$$

$$\mathcal{E}_{\mu\nu}=\frac{1}{4\pi}\left[F_{\mu\lambda}F_{\nu}{}^{\lambda}-\frac{1}{4}g_{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma}\right], \tag{26}$$

$$\frac{d\psi}{dr}=\frac{1}{2r(2f_R+rR'f_{RR})}\left\{r^2e^{2\lambda}\left[16\pi\left(p-\frac{q^2}{8\pi r^4}\right)+f-Rf_R\right]+2f_R\left(e^{2\lambda}-1\right)-4rR'f_{RR}\right\},$$

$$\frac{d\lambda}{dr}=\frac{1}{2r(2f_R+rR'f_{RR})}\left\{2f_R\left(1-e^{2\lambda}\right)+\frac{r^2e^{2\lambda}}{3}\left[16\pi\left(2\rho+3p+\frac{3q^2}{8\pi r^4}\right)+Rf_R+f\right]\right.$$
$$\left.+\frac{rR'f_{RR}}{f_R}\left[2f_R\left(1-e^{2\lambda}\right)+\frac{r^2e^{2\lambda}}{3}\left(16\pi\rho+\frac{6q^2}{r^4}+2Rf_R-f\right)+2rR'f_{RR}\right]\right\},$$

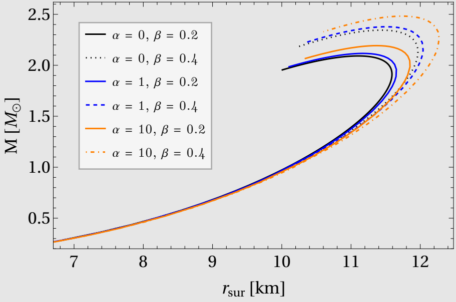
$$\frac{d^2R}{dr^2}=\frac{1}{3f_{RR}}\left\{e^{2\lambda}\left[8\pi(-\rho+3p)+2f-Rf_R\right]-3R'^2f_{RRR}\right\}+\left(\lambda'-\psi'-\frac{2}{r}\right)R',$$

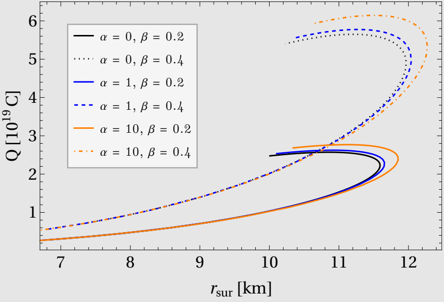
$$\frac{dp}{dr}=-(\rho+p)\psi'+\frac{qq'}{4\pi r^4},$$

$$\frac{dq}{dr}=4\pi r^2\rho_{\rm ch}e^{\lambda},$$

Numerical results

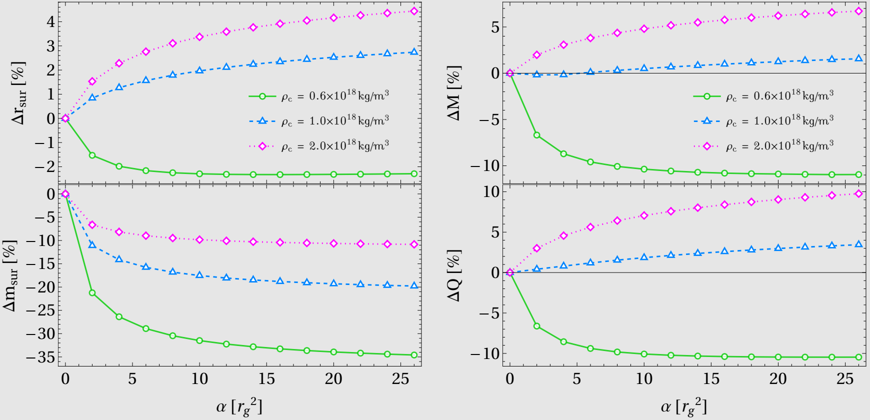
- Specific function: $f(R) = R + \alpha R^2$
- MIT bag model EoS: $p = \frac{1}{3}(\rho - 4B)$
- Charge profile: $\rho_{\text{ch}} = \beta\rho$





Both parameters allow an increase in the maximum-mass values, however, the greater effect is obtained by varying β. Furthermore, the most relevant changes in the total charge due to the quadratic term occur in the high-radius region.

Relative deviation as a function of the free parameter α for three values of central density and fixed $\beta = 0.2$: $\Delta = \left.\frac{\Upsilon_{f(R)} - \Upsilon_{\rm GR}}{\Upsilon_{\rm GR}}\right|_{\rho_c}$



The mass measured at the surface always decreases regardless of ρ_c and variations can reach up to 35%. We also observe that the asymptotic mass and the total charge have a similar behavior, reaching deviations of up to 10%.

Conclusions

- ⋆ In this work we have investigated the global physical properties of compact stars within the context of metric power-law $f(R)$ gravity. In particular, we considered the gravitational action $f(R) = R^{1+\epsilon}$ in order to explore small deviations from the usual Einstein-Hilbert action for $|\epsilon| \ll 1$.
- ⋆ Under a non-perturbative formulation, we have derive the modified TOV equations and studied the equilibrium structure of compact stars in the presence of anisotropy. We have adopted the anisotropy profile suggested by Horvat and collaborators, where appears a dimensionless parameter β which controls the degree of anisotropy within the compact star.
- ⋆ We have found that the main effect of the parameter ϵ is to increase the total gravitational mass throughout the range of central energy densities. In addition, the relevant changes due to the anisotropic pressure emerge in the high-mass region (close to the maximum-mass point), while the variations are negligible in the low-mass branch.

- ⋆ We investigated the structure of (strange) quark stars within the R^2-gravity framework, also known as the Starobinsky model, using an equation of state for quark matter obtained from cold and dense perturbative QCD. We have analyzed the consequences of the extra term αR^2 on the properties of quark stars.
- ⋆ We have found that the mass parameter at the stellar surface, $m_{\rm sur}$, suffers more significant variations when compared to GR, whereas the deviations of the asymptotic mass, M, are almost negligible for small masses but more significant in the maximum-mass region.
- ⋆ We have explored the effect of electric charge on the global properties of compact stars within the context of fourth-order $f(R)$ theories of gravity.
- ⋆ According to the total charge versus radius relation, the most substantial changes due to the Starobinsky term occur in the high-radius region. Nevertheless, it is worth emphasizing that the largest changes in radius, mass, and charge are due to the charge parameter β.


Thanks for your attention