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Quantum mechanics quasi-exactly solvable problems.

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- In quantum mechanics, a QES spectral problem is one for which it is not possible to obtain the complete energy spectrum analytically. two, and higher dimensions.
- In the literature, there exist three direct methods to find the solutions of QES quantum-mechanical problems:
  i) use of a simple polynomial expansion,
  (ii) transforming the Schrödinger equation into a known equation, like the CHE, and
  (iii) using Lie algebras.
- The first method is used to show, in a very simple way, that for some QES problems, the wave solutions and the potentials share an intimate relation that preclude to fix the latter and allow for a recursive method to generate the complete spectrum.
- Secondly, problems where the Heun equation is used to find the spectra have been shown to be in the QES class, and in our case we find that the solutions found in the first method turn out to be the same.

- On the other hand, the use of Lie algebraic methods have been used to reveal the existence of hidden algebraic structures in problems which do not show any hidden symmetry properties, a feature which according to Turbiner.
- Turbiner was the first to give a detailed list of potentials for QES problems, while González-López and collaborators further developed these techniques to a larger list of potentials.
- However, being simpler, the polynomial expansion based on the Bethe Ansatz method and the CHE transformation that we also use here have been the common tools in many more recent studies on QES spectral problems.
- On the other hand, hyperbolic and trigonometric type potentials are used in molecular physics and quantum chemistry, modeling inter-atomic and inter-molecular forces, ranging from Razavy, Pöschl-Teller, Rosen-Morse, and Scarf potentials to their modified counterparts.
- In quantum chemistry, the area of IR-spectroscopy is particularly interesting, since double-well potentials (DWP) could describe the ammonia molecule (NH_3), chromous acid (CrOOH) and potassium dihydrogen phosphate (KH_2PO_4) experimental data.

- Let us consider the spectral problem for the Schrodinger equation

$$-\frac{d^2\Psi(x)}{dx^2} + V(x)\Psi(x) = E\Psi(x) \tag{1}$$

where $V(x)$ is the potential given in the equations below.

- We work on the solutions for the hyperbolic potential

$$V(x) = 4\gamma^2 \cosh^4(x) + V_1(\gamma,\eta)\cosh^2(x) + \eta\,(\eta-1)\tanh^2(x), \tag{2}$$

- And its trigonometric counterpart,

$$U(x) = -4\gamma^2 \cos^4(x) - V_1(\gamma,\eta)\cos^2(x) + \eta\,(\eta-1)\tan^2(x) \tag{3}$$

- The anti-isospectral transformation $x \to ix$, relates theses two potentials with following relation $U(x;\gamma,\eta) = -V(ix;\gamma,\eta)$.
- Also relates the part of the spectrum found for the hyperbolic potential with the opposite sign part of the spectrum of the trigonometric potential.

- Let us consider the spectral problem for the 1D Schrodinger equation with $V(x)$ given in equation 2 or 3.
- We first find solutions using a polynomial expansion of $\Psi(x)$.
- The form of the the polynomial expansion is

$$f(z) = f_0 \prod_{i=1}^{N} (z - z_{N,i}) \tag{4}$$

  where **N**, is the order of the polynomial.
- These problems belong to the class of QES potentials, and the solvable part of the energy spectrum is found to depend on the order of the polynomial, **N**.
- Have two classes of solution: even and odd solutions as trial functions, called here trial functions TF.

- We first look for even solutions in the case of the hyperbolic potential (2), with eigenfunctions

$$\Psi_1(x) = e^{-\gamma \cosh^2(x)} \cosh^{\eta}(x) f(x). \tag{5}$$

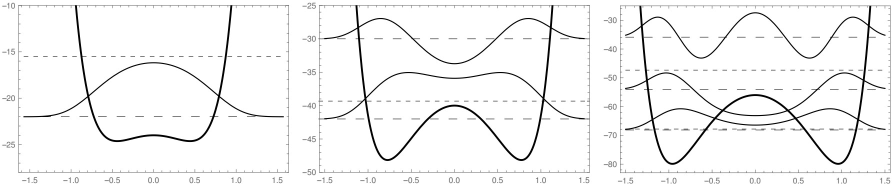
- By using the change of variable $z = \cosh^2(x)$, we can immediately find a CHE equation,

$$\frac{d^2 f}{dz} + \left[-2\gamma + \frac{\eta + 1/2}{z} + \frac{1/2}{z-1}\right] \frac{df}{dz} + \left[\frac{-1/4\left(E + \eta + 2\gamma\left(2\eta + 1\right)\right)}{z} + \frac{1/4\left(E + \eta - V_1 - 2\gamma\left(2\gamma + 1\right)\right)}{z-1}\right] f = 0 \tag{6}$$

- We look for the polynomial expansion solution, using the polynomial expansion given in the equation 4.

- The only solutions found depend on the order of the polynomial.
- For $\mathbf{N=0}$, we can only find one energy value, $E=-2\gamma\left(2\eta+1\right)-\eta$;
- For $\mathbf{N=1}$, we find two eigenvalues, $E=-3\eta-6\gamma-2-4\gamma\eta\pm2\sqrt{(\eta+1)^2+4\gamma(\gamma-\eta)}$, and the polynomial roots $z_{1,i}=-\frac{2(2\eta+1)}{E+2\gamma(2\eta+1)+\eta}$.
- For $\mathbf{N=2}$ we obtain a third order equation for the energy eigenvalues, which we only solve numerically.
- In Table 1 we give a summary of the energy eigenvalues found analytically and numerically, when the parameters are $\gamma=2$ and $\eta=2$.
- The polynomial roots are, for N=2, the three pairs of roots are $(z_{2,1},z_{2,2})=(0.294,\,0.823)$, $(0.388,\,1.612)$, and $(1.124,\,2.008)$.
- In all cases, the coefficient V_1 is found to be $V_1=-4\gamma(2N+1+\gamma+\eta)$, whose dependence on N forbids scaling solutions of different polynomial expansion orders.
- Also, the polynomial order gives the number of roots in each case, and that determines the number of eigenfunctions to be $N+1$ solutions.

Polynomial expansion
Hyperbolic potential – TF1



(a) N = 0 (b) N = 1 (c) N = 2

Figure 1: Potential functions and the TF1 eigenfunctions for $\mathbf{N} = \mathbf{0}$, $\mathbf{1}$ and $\mathbf{2}$, and $\gamma = \eta = 2$.

- We now look for odd solutions of the type

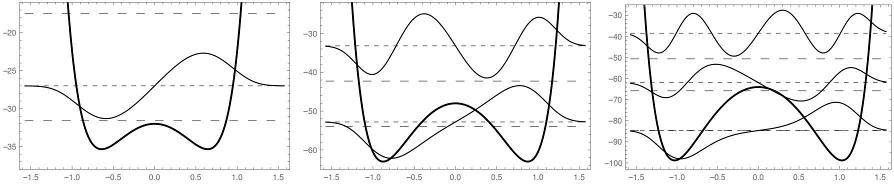
$$\Psi_2(x) = e^{-\gamma \cosh^2(x)} \cosh^\eta(x) \sinh(x) f(x). \tag{7}$$

- We find another CHE in terms of the variable $z = \cosh^2(x)$,

$$\frac{d^2 f}{dz^2} + \left[-2\gamma + \frac{\eta + 1/2}{z} + \frac{3/2}{z-1}\right] \frac{df}{dz} + \left[\frac{-1/4\,(E + 2\gamma\,(2\eta+1) + 3\eta + 1)}{z} + \frac{1/4\,(E - V_1 - 2\gamma\,(2\gamma+3) + 3\eta + 1)}{z-1}\right] f = 0 \tag{8}$$

- Again, using the polynomial expansion in the equation 4, we find that here $V_1 = -4\gamma(2N + 2 + \gamma + \eta)$.
- As of the eigenvalues, for $\mathbf{N = 0}$, we get $E = -1 - 3\eta - 2\gamma\,(2\eta + 1)$.
- For $\mathbf{N = 1}$, we find the eigenvalues $E = -6\gamma - 5\eta - 4\gamma\eta - 5 \pm 2\sqrt{(\eta+2)^2 + 4\gamma(\gamma - \eta + 1)}$, and the roots $z_{1,1} = -\frac{2(2\eta+1)}{E + 2\gamma(2\eta+1) + 3\eta + 1}$.
- For the case $\mathbf{N = 2}$, with $\gamma = 2$ and $\eta = 2$, the three eigenvalues are $E_1 = -84.635$, $E_3 = -61.915$ and $E_5 = -38.449$, and the pairs of roots are $(z_{2,1}, z_{2,2}) = (0.235, 0.335)$, $(2.417, 0.663)$, and $(1.983, 1.368)$.

Polynomial expansion
Hyperbolic potential – TF2



(a) $\mathrm{N} = 0$ (b) $\mathrm{N} = 1$ (c) $\mathrm{N} = 2$

Figure 2: For the TF2 the potential and wave functions for $\mathbf{N} = \mathbf{0}$, $\mathbf{1}$ and $\mathbf{2}$, when γ, η=2.

	TF1			TF2		
	N = 0	N = 1	N = 2	N = 0	N = 1	N = 2
E_0	**-22.000**	**-42.000**	**-68.124**	-31.606	-53.922	-84.704
E_1	-15.489	-39.323	-67.801	**-27.000**	**-52.798**	**-84.635**
E_2	-5.186	**-30.000**	**-54.000**	-17.502	-42.265	-65.806
E_3	7.489	-19.350	-47.331	-5.773	**-33.202**	**-61.915**
E_4	22.215	-6.315	**-35.875**	8.108	-21.011	-50.642
E_5	38.772	8.674	-22.557	23.880	-6.822	**-38.449**
E_6	57.008	25.435	-7.300	41.377	9.198	-24.001
E_7	76.809	43.837	9.690	60.477	26.900	-7.753
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Table 1: Exact (bold type) and numerical eigenvalues for the hyperbolic potential (2), for even (TF1) and odd (TF2) eigenfunctions, with $\mathbf{N} = \mathbf{0}$, $\mathbf{1}$ and $\mathbf{2}$.

- Let us consider here the case of

$$\Psi_3(x) = e^{-\gamma \cosh^2(x)} \operatorname{sech}^{\eta-1}(x) f(x). \tag{9}$$

- The exponent of $\operatorname{sech}(x)$ is set for convenience.
- Using $z = \cosh^2(x)$, we arrive at the CHE equation

$$\frac{d^2 f}{dz^2} + \left[-2\gamma + \frac{-\eta + 3/2}{z} + \frac{1/2}{z-1}\right]\frac{df}{dz} + \left[\frac{-1/4\,(E + 2\gamma\,(3 - 2\eta) - \eta + 1)}{z} + \frac{1/4\,(E - V_1 - 2\gamma\,(2\gamma + 1) - \eta + 1)}{z-1}\right] f = 0 \tag{10}$$

- The polynomial solutions of order N render the factor $V_1 = -4\gamma\,(2N + 1 + \gamma + \eta)$.
- The analytical and numerically found eigenvalues are given in Table 2, for **N** = **0**, **1** and **2**.

- As said above, we may consider a fourth case, of odd eigenfunctions

$$\Psi_4(x) = e^{-\gamma \cosh^2(x)} \operatorname{sech}^{\eta-1}(x) \sinh(x) f(x) \tag{11}$$

- The CHE

$$\frac{d^2 f}{dz^2} + \left[-2\gamma + \frac{-\eta + 3/2}{z} + \frac{3/2}{z-1}\right] \frac{df}{dz} + \left[\frac{-1/4\left(E + 2\gamma\left(3 - 2\eta\right) - 3\eta + 4\right)}{z} + \frac{1/4\left(E - V_1 - 2\gamma\left(2\gamma + 3\right) - 3\eta + 4\right)}{z-1}\right] f = 0 \tag{12}$$

- The analytical and numerically found eigenvalues are given in Table 2, for **N** $= \mathbf{0}$, $\mathbf{1}$ and $\mathbf{2}$.

	TF3			TF4		
	N = 0	N = 1	N = 2	N = 0	N = 1	N = 2
E_0	**5.000**	**-12.798**	**-31.606**	−3.826	−22.000	−42.000
E_1	16.250	−4.544	−27.000	**6.000**	**-15.489**	**-39.323**
E_2	29.800	**6.798**	**-17.502**	18.447	−5.186	−30.000
E_3	45.329	20.417	−5.773	33.021	**7.489**	**-19.350**
E_4	62.635	35.998	**8.108**	49.464	22.215	−6.315
E_5	81.579	53.346	23.880	67.610	38.772	**8.674**
E_6	102.057	72.325	41.377	87.337	57.008	25.435
E_7	123.986	92.833	60.477	108.555	76.809	43.837
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Table 2: Exact (bold type) and numerical eigenvalues for the hyperbolic potential (2), for even (TF3) and odd (TF4) eigenfunctions, with **N** = **0**, **1** and **2**.

- The trigonometric potential (3), can be obtained from the hypergeometric one by equation (2), via the anti-isospectral transformation $x \rightarrow ix$.
- This in turn implies that the trigonometric eigenvalues should be opposite sign of the hypergeometric cases.
- Now we can see that trial functions type 1 and 2 can be transformed into regular solutions of the potential (3).
- While trial functions type 3 and 4 would not render square integrable eigenfunctions due to the $\sec(x)$ term appearing there.
- Therefore, trial functions 1 and 2, where $\eta > 0$, are the only two which can be used in the trigonometric case. Here we shall only summarize the results for these functions.

- For this case, for the eigenfunction type

$$\Phi_1 = e^{-\gamma \cos^2(x)} \cos^\eta(x) f(z) \tag{13}$$

with $z = \cos^2(x)$.

- We find the CHE

$$\frac{d^2 f}{dz^2} + \left[-2\gamma + \frac{\eta + 1/2}{z} + \frac{1/2}{z-1}\right]\frac{df}{dz} + \left[\frac{1/4\left(E - \eta - 2\gamma\left(2\eta + 1\right)\right)}{z} - \frac{1/4\left(E - \eta - V_1 + 2\gamma\left(2\gamma + 1\right)\right)}{z-1}\right] f = 0 \tag{14}$$

and $V_1 = 4\gamma\left(2N + 1 + \eta + \gamma\right)$, same as in the hyperbolic potential.

- For $\mathbf{N} = \mathbf{0}$ the energy is $E_0 = \eta + 2\gamma(2\eta + 1)$.
- For $\mathbf{N} = \mathbf{1}$ we find $E = 3\eta + 6\gamma + 4\gamma\eta + 2 \pm \sqrt{\left(\eta + 1\right)^2 + 4\gamma\left(\gamma - \eta\right)}$, with $z_{1,1} = -\frac{2(2\eta+1)}{-E+2\gamma(2\eta+1)+\eta}$.

- For the odd eigenfunctions, we propose

$$\Phi_2 = e^{-\gamma \cos^2(x)} \cos^{\eta}(x) \sin(x) f(x) \tag{15}$$

we find that $V_1 = 4\gamma (2N + 2 + \eta + \gamma)$.

- The CHE

$$\frac{d^2 f}{dz^2} + \left[-2\gamma + \frac{\eta + 1/2}{z} + \frac{3/2}{z-1}\right] \frac{df}{dz} + \left[\frac{1/4 (E - 1 - 3\eta - 2\gamma (2\eta + 1))}{z} - \frac{1/4 (E - 1 - 3\eta - V_1 + 2\gamma (2\gamma + 3))}{z-1}\right] f = 0 \tag{16}$$

- For $\mathbf{N} = \mathbf{0}$ the energy is $E = 1 + 3\eta + 2\gamma (2\eta + 1)$.
- For $\mathbf{N} = \mathbf{1}$ $E = 5 + 5\eta + 6\gamma + 4\gamma\eta \pm 2\sqrt{(\eta + 2)^2 + 4\gamma (\gamma - \eta + 1)}$, with $z_{1,1} = -\frac{2(2\eta+1)}{-E+2\gamma(2\eta+1)+3\eta+1}$.
- The analytical and numerically found eigenvalues are displayed in Table 3, for the cases with $\gamma = 2$ and $\eta = 2$, and for $\mathbf{N} = \mathbf{0}$, $\mathbf{1}$ and $\mathbf{2}$.
- Comparing to the results in Tables 1 and 3 , we can notice the reverse sign eigenvalues for the analytically found portion of the spectrum.

	TF1			TF2		
	N = 0	N = 1	N = 2	N = 0	N = 1	N = 2
E_0	**22.000**	**30.000**	**35.875**	26.400	33.098	38.429
E_1	23.394	30.247	35.921	**27.000**	**33.202**	**38.449**
E_2	30.368	**42.000**	**54.000**	35.979	48.088	59.580
E_3	38.656	48.088	57.421	43.351	**52.798**	**61.915**
E_4	49.195	58.331	**68.124**	53.703	63.119	73.404
E_5	61.911	70.764	79.935	66.299	75.310	**84.635**
E_6	76.716	85.383	94.290	81.020	89.806	98.841
E_7	93.576	102.113	110.837	97.822	106.451	115.273
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Table 3: Exact (bold type) and numerical eigenvalues for the trigonometric potential (3), for even (TF1) and odd (TF2) eigenfunctions, with $\mathbf{N} = \mathbf{0}$, $\mathbf{1}$ and $\mathbf{2}$.

- Now, we shall compare the analytical solutions found in the section of polynomial expansion, with those found matching all coefficients in equations (6,8,10,12), for the hyperbolic cases, with the coefficients in the CHE.
- The CHE

$$\frac{d^2H(z)}{dz^2} + \left(\alpha + \frac{1+\beta}{z} + \frac{1+\gamma^*}{z-1}\right)\frac{dH(z)}{dz} + \left(\frac{\mu}{z} + \frac{\nu}{z-1}\right)H(z) = 0 \quad (17)$$

- And posses the solution

$$H_C\left(\alpha, \beta, \gamma^*, \delta, \eta^*, z\right) = \sum_{i=0}^{\infty} v_N\left(\alpha, \beta, \gamma^*, \delta, \eta^*\right) z^i \quad (18)$$

where

$$\delta = \mu + \nu - \frac{\alpha}{2}\left(\beta + \gamma + 2\right) \quad (19)$$

$$\eta^* = \frac{\alpha}{2}\left(\beta + 1\right) - \mu - \frac{1}{2}\left(\beta + \gamma + \beta\gamma\right) \quad (20)$$

Matching terms in the CHE
Theory

- The coefficients v_N are given by the three-term recurrence relation

$$A_N v_N = B_N v_{N-1} + C_N v_{N-2}, \quad \text{with initial conditions} \quad v_{-1} = 0,\ v_0 = 1 \tag{21}$$

where

$$A_N = 1 + \frac{\beta}{N} \tag{22}$$

$$B_N = 1 + \frac{1}{N}(\beta + \gamma - \alpha - 1) + \frac{1}{N^2}\left\{\eta^* - \frac{1}{2}(\beta + \gamma - \alpha) - \frac{\alpha\beta}{2} + \frac{\beta\gamma^*}{2}\right\} \tag{23}$$

$$C_N = \frac{\alpha}{N^2}\left(\frac{\delta}{\alpha} + \frac{\beta + \gamma^*}{2} + N - 1\right) \tag{24}$$

- To reduce a confluent Heun function to a confluent Heun polynomial of degree N, we need two termination conditions which are

$$\begin{aligned} \mu + \nu + N\alpha &= 0 \\ \Delta_{N+1}(\mu) &= 0 \end{aligned} \tag{25}$$

- The second termination condition, $\Delta_{N+1}(\mu) = 0$, can be represented as

$$\begin{vmatrix} \mu - q_1 & (1+\beta) & 0 & \dots & 0 & 0 & 0 \\ N\alpha & \mu - q_2 + \alpha & 2(2+\beta) & \dots & 0 & 0 & 0 \\ 0 & (N-1)\alpha & \mu - q_3 + 2\alpha & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \mu - q_{N-1} + (N-2)\alpha & (N-1)(N-1+\beta) & 0 \\ 0 & 0 & 0 & \dots & 2\alpha & \mu - q_N + (N-1)\alpha & N(N+\beta) \\ 0 & 0 & 0 & \dots & 0 & \alpha & \mu - q_{N+1} + N\alpha \end{vmatrix} = 0 \tag{26}$$

where $q_n = (n-1)(n+\beta+\gamma^*)$.

- For comparison, we need to set $z = \cosh^2(x)$.

- For TF1, we compare the coefficients in eq.(4) to those of the Heun eq.(17), to obtain

$$\alpha = -2\gamma,\ \beta = \eta - \frac{1}{2},\ \gamma^* = -\frac{1}{2},\ \mu = -\frac{1}{4}\left(E + \eta + 2\gamma\left(2\eta + 1\right)\right),\ \text{and}$$
$$\nu = \frac{1}{4}\left(E + \eta - V_1 - 2\gamma\left(2\gamma + 1\right)\right)$$

- With the use of equations (19,20) and (25), we find that $\delta = \gamma\left(2N + \eta + 1\right)$, $\eta^* = \frac{1}{8}\left(2E + 3\right)$ and $V_1 = -4\gamma\left(\gamma + \eta + 2N + 1\right)$.
- The wave function is

$$\psi(x) = e^{-\gamma \cosh^2(x)} \cosh^{\eta}(x) H_C\left(\alpha, \beta, \gamma^*, \delta, \eta^*\right) \tag{27}$$

- For example, in the case N $= 0$, $\Delta_1 = \mu - q_1 = 0$, $q_1 = 0$, and $\mu = 0$. Therefore $E = -\eta - 2\gamma\left(2\eta + 1\right)$, as we found above.
- For **N = 1**, we find that

$$\Delta_2 = \begin{vmatrix} \mu - q_1 & 1 + \beta \\ \alpha & \mu - q_2 + \alpha \end{vmatrix} = 0, \quad \text{with } q_1 = 0 \text{ and } q_2 = 2 + \beta + \gamma^*$$

- Then,

$$\mu^2 - (2+\beta+\gamma^* - \alpha)\,\mu - \alpha\,(1+\beta) = 0, \text{ and } \mu = -\frac{1}{4}\left(E+\eta+2\gamma\,(2\eta+1)\right)$$

rendering that $E=-(3\eta+2)(\gamma+1)-\gamma(\eta+4)\pm 2\sqrt{(\eta+1)^2+4\gamma(\gamma-\eta)}$, and the expansion coefficient $v_1 = \frac{E+2\gamma+\eta+4\gamma\eta}{2(2\eta+1)}$, which is just $-1/z_{1,1}$ in the polynomial expansion in TF1.

- Meanwhile, for $\mathbf{N=2}$ we obtain the 3×3 matrix

$$\Delta_3 = \begin{vmatrix} \mu - q_1 & 1+\beta & 0 \\ 2\alpha & \mu - q_2 + \alpha & 2\,(2+\beta) \\ 0 & \alpha & \mu - q_3 + 2\alpha \end{vmatrix} = 0, \text{ where } q_1=0,\ q_2=2+\beta+\gamma^* \text{ and } q_3=2(3+\beta+\gamma^*)$$

- The determinant gives

$$\mu^3 - (3\eta+6\gamma+5)\,\mu^2 - 2\left((\eta+2)(\eta+1) - \gamma\,(4\gamma-3)\right)\mu - 8\gamma\,(2\gamma+\eta+2) = 0,$$

where $\mu = -\frac{1}{4}\left(E+\eta+2\gamma\,(2\eta+1)\right)$.

- The numerical eigenvalues correspond to those found in TF1.

- For the odd functions (10), matching terms with those of the Heun eq.(18), we obtain

$$\alpha = -2\gamma,\ \beta = \eta - \frac{1}{2},\ \gamma^* = \frac{1}{2},\ \mu = -\frac{1}{4}\left(E + 3\eta + 1 + 2\gamma\left(2\eta + 1\right)\right),\ \text{and}$$
$$\nu = \frac{1}{4}\left(E + 3\eta + 1 - V_1 - 2\gamma\left(2\gamma + 3\right)\right)$$

- Hence, we find that $\delta = \gamma\left(\eta + 2 + 2N\right)$, $\eta^* = \frac{1}{8}\left(2E + 3\right)$ and $V_1 = -4\gamma\left(\gamma + \eta + 2N + 2\right)$. As for the wave eigenvalues and eigenfunctions, when N = 0, we find $E = -1 - 3\eta - 2\gamma\left(2\eta + 1\right)$.
- For the case **N = 1**, the determinant is

$$\Delta_2 = \begin{vmatrix} \mu - q_1 & 1 + \beta \\ \alpha & \mu - q_2 + \alpha \end{vmatrix} = 0, \quad \text{where } q_1 = 0 \text{ and } q_2 = 2 + \beta + \gamma^*$$

then,

$$\mu^2 - \left(2 + \beta + \gamma^* - \alpha\right)\mu - \alpha\left(1 + \beta\right) = 0,$$

where $\mu = -\frac{1}{4}\left(E + 1 + 3\eta + 2\gamma\left(2\eta + 1\right)\right)$

- Rendering $E=-5-5\eta-6\gamma-4\gamma\eta\pm2\sqrt{(\eta+2)^2+4\gamma(\gamma-\eta+1)}$. The expansion coefficient $v_1=\frac{E+2\gamma+3\eta+1+4\gamma\eta}{2(2\eta+1)}$ is the negative inverse of the root found in TF2, and the eigenvalues also coincide with those found there.
- As a final example, for N $=2$, we obtain the 3×3 matrix

$$\Delta_3=\begin{vmatrix}\mu-q_1 & 1+\beta & 0\\ 2\alpha & \mu-q_2+\alpha & 2\left(2+\beta\right)\\ 0 & \alpha & \mu-q_3+2\alpha\end{vmatrix}=0, \text{ where } q_1=0,\ q_2=2+\beta+\gamma^* \text{ and } q_3=2(3+\beta+\gamma^*)$$

giving

$$\mu^3-\left(3\eta+6\gamma+8\right)\mu^2+\left(12+10\eta+2\eta^2+28\gamma+24\eta\gamma\right)\mu-8\gamma\left(2\gamma+\eta+3\right)=0,$$

where $\mu=-\frac{1}{4}\left(E+1+3\eta+2\gamma\left(2\eta+1\right)\right)$

- In this case, eigenvalues are found numerically, but the expansion coeeficients are found as $v_1=\frac{E+2\gamma+3\eta+1+4\gamma\eta}{2(2\eta+1)}$, and $v_2=\frac{(E+7\eta+9+2\gamma(2\eta+5))(E+3\eta+1+2\gamma(2\eta+1))+32\gamma(2\eta+1)}{8(2\eta+1)(2\eta+3)}$.

- In comparing eq.(14) to the CHE (17), we obtain the followings values

$$\alpha = -2\gamma,\ \beta = \eta - \frac{1}{2},\ \gamma^* = -\frac{1}{2},\ \mu = \frac{1}{4}\left(E - \eta - 2\gamma\left(2\eta + 1\right)\right),\ \text{and}$$
$$\nu = -\frac{1}{4}\left(E - \eta - V_1 + 2\gamma\left(2\gamma + 1\right)\right)$$

- And using μ and ν from eqs.(19), (20) and (25), we obtain $\delta = \gamma\left(\eta + 1 + 2N\right)$, $\eta^* = \frac{1}{8}\left(-2E + 3\right)$ and $V_1 = 4\gamma\left(\gamma + \eta + 2N + 1\right)$, as before.
- As of the example cases worked above, for $\mathrm{N} = 0$, then we find $\Delta_1 = \mu - q_1 = 0$ and $q_1 = 0$; therefore $\mu = 0$, and so $E = \eta + 2\gamma\left(2\eta + 1\right)$.
- For $\mathbf{N} = \mathbf{1}$, with $q_1 = 0$ and $q_2 = 2 + \beta + \gamma^*$, the corresponding determinant is $\mu^2 - \left(2 + \beta + \gamma^* - \alpha\right)\mu - \alpha\left(1 + \beta\right) = 0$, where $\mu = \frac{1}{4}\left(E - \eta - 2\gamma\left(2\eta + 1\right)\right)$, rendering $E=3\eta+6\gamma+2+4\gamma\eta\pm2\sqrt{(\eta+1)^2+4\gamma(\gamma-\eta)}$, and the expansion coefficient $v_1 = \frac{-E+2\gamma+\eta+4\gamma\eta}{2(2\eta+1)}$.
- Finally, for $\mathrm{N} = 2$, the expansion coefficients are $v_1 = \frac{-E+2\gamma+\eta+4\gamma\eta}{2(2\eta+1)}$, and $v_2 = \frac{(-E+10\gamma+5\eta+4(\gamma\eta+1))(-E+\eta+2\gamma(2\eta+1))+32\gamma(2\eta+1)}{8(2\eta+1)(2\eta+3)}$.

- For odd solutions, comparing eq.(16) to eq.(17) we find that

$$\alpha = -2\gamma,\ \beta = \eta - \frac{1}{2},\ \gamma^* = \frac{1}{2},\ \mu = \frac{1}{4}\left(E - 1 - 3\eta - 2\gamma\left(2\eta + 1\right)\right),\ \text{and}$$
$$\nu = -\frac{1}{4}\left(E - 1 - 3\eta - V_1 + 2\gamma\left(2\gamma + 3\right)\right)$$

- Together with $\delta = \gamma\left(\eta + 2 + 2N\right)$, $\eta^* = \frac{1}{8}\left(-2E + 3\right)$, we get $V_1 = 4\gamma\left(\gamma + \eta + 2N + 2\right)$.
- Continuing our examples, for N $= 0$ we find $E = 1 + 3\eta + 2\gamma\left(2\eta + 1\right)$.
- For N $= 1$ we find $E = 5 + 5\eta + 6\gamma + 4\gamma\eta \pm 2\sqrt{\left(\eta + 2\right)^2 + 4\gamma\left(\gamma - \eta + 1\right)}$, with $v_1 = \frac{-E+2\gamma+3\eta+4\gamma\eta+1}{2(2\eta+1)}$.
- For N $= 2$, we find the expansion coefficients $v_1 = \frac{-E+2\gamma+3\eta+1+4\gamma\eta}{2(2\eta+1)}$ and $v_2 = \frac{(-E+7\eta+9+10\gamma+4\gamma\eta)(-E+3\eta+2\gamma(2\eta+1)+1)+32\gamma(2\eta+1)}{8(2\eta+1)(2\eta+3)}$, and solve for the energies numerically.

- We now come to solutions found using the Lie algebraic procedure. For this matter, we shall follow the work of Finkel et al.
- We begin by writing the potential function (2)

$$V(x) = 4\gamma^2 \cosh^4(x) - 4\gamma\,(\eta + \gamma + M)\cosh^2(x) + \eta\,(\eta - 1)\tanh^2(x) \quad (28)$$

where $V_1 = -4\gamma\,(\eta + \gamma + M)$, and M for all four trial functions in polynomial expansion is given in Table 4.

- We utilize a QES potential function given in Finkel *et al*

$$V(x) = A\cosh^2(\sqrt{\nu}x) + B\cosh(\sqrt{\nu}x) + C\coth(\sqrt{\nu}x)\operatorname{csch}(\sqrt{\nu}x) + D\operatorname{csch}^2(\sqrt{\nu}x) \quad (29)$$

- With value of $\nu = 4$, we match the equations (28) and (29).
- We obtain the constants that produce the subsequent values $A = \gamma^2$, $B = -2\gamma\,(\eta + M)$ and $D = -C = 2\eta\,(2\eta - 1)$ and some constant which is unnecessary during our procedure.
- With the advice of these constants we obtain the gauge transformation for this hyperbolic potential

	M	σ	Eigenfunctions
TF1	2N + 1	2η	$\Psi_1 = e^{-\gamma \cosh^2(x)} \cosh^{\eta}(x)\, f(\cosh^2(x))$
TF2	2N + 2	$2\eta + 1$	$\Psi_2 = e^{-\gamma \cosh^2(x)} \cosh^{\eta}(x) \sinh(x)\, f(\cosh^2(x))$
TF3	2N - 2η + 2	1	$\Psi_3 = e^{-\gamma \cosh^2(x)} \operatorname{sech}^{\eta-1}(x)\, f(\cosh^2(x))$
TF4	2N - 2η + 3	2	$\Psi_4 = e^{-\gamma \cosh^2(x)} \operatorname{sech}^{\eta-1}(x) \sinh(x)\, f(\cosh^2(x))$

Table 4: The four types of eigenfunction for the hyperbolic potential (2). The parameter σ appears in the gauge transformation function $\hat{\mu}(z)$.

- Setting $\psi(z) = \hat{\mu}(z)\hat{\chi}(z)$, and using the gauge transformation function in terms of the variable $z = \cosh(2x)$

$$\hat{\mu}(z) = (z-1)^{\frac{1}{4}\left(-\eta+\sigma+\frac{\eta(\eta-1)}{-\eta+\sigma-1}\right)}(z+1)^{\frac{1}{4}\left(-\eta+\sigma-\frac{\eta(\eta-1)}{-\eta+\sigma-1}\right)}e^{-\frac{\gamma}{2}z} \tag{30}$$

- The $\mathfrak{sl}(2,\mathbb{R})$ operators

$$J_- = \partial_z, \; J_0 = z\partial_z - \frac{N}{2}, \; J_+ = z^2\partial_z - Nz$$

- We find that the gauge Hamiltonian may be written as

$$\hat{H}_g(z) = -4J_0^2+4J_-^2+4\gamma J_+ +4\left(-\sigma+\eta-N\right)J_0+4\left(-\gamma+\frac{\eta\left(\eta-1\right)}{-\eta+\sigma-1}\right)J_- + c_* \tag{31}$$

where $c_* =$
$N^2+2N\left(\sigma-\eta\right)+4\gamma N+2\gamma\left(-\eta+\sigma+1\right)-\eta\left(\eta+1\right)+\frac{2\gamma\eta(\eta-1)}{-\eta+\sigma-1}+\left(\eta-\sigma\right)^2$.

- Following the Lie algebraic method, we only need to look for the orthogonal polynomial solutions

$$\hat{\chi}(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{\sigma}{2} + \frac{\eta(2\eta-\sigma)}{2(-1-\eta+\sigma)} + k\right)!}{2^k\left(\sigma + \frac{\eta(2\eta-\sigma)}{-1-\eta+\sigma} + 2k\right)!} \frac{(z+1)^k}{k\,!} \hat{P}_k \tag{32}$$

- where the three-term recurrence relation

$$\hat{P}_{k+1} = (E - b_k)\hat{P}_k - a_k\hat{P}_{k-1}, \quad k \geq 0 \tag{33}$$

is satisfied for

$$a_k = 16\gamma k\,(k - N - 1)\left(2k - 1 + \sigma - \eta + \frac{\eta\,(\eta-1)}{-1-\eta+\sigma}\right)$$

$$b_k = -4k\,(\sigma - \eta + k + 2\gamma) - 2\gamma\,(-\eta + \sigma + 1) + \eta\,(\eta - 1) - \frac{2\gamma\eta\,(\eta-1)}{-1-\eta+\sigma} - (\eta-\sigma)^2$$

Eigenvalues are found when we set $\hat{P}_{k+1} = 0$.

- Two sets of solutions are found this way, which we divided in two more categories to include negative η in Table (4).
- Finally, the eigenfunctions are given by

$$\Psi(z) = (z+1)^{\frac{1}{4}\left(-\eta+\sigma+\frac{\eta(\eta-1)}{-\eta+\sigma-1}\right)}(z-1)^{\frac{1}{4}\left(-\eta+\sigma-\frac{\eta(\eta-1)}{-\eta+\sigma-1}\right)}e^{-\frac{\gamma}{2}z} \sum_{k=0}^{\infty} \frac{\left(\frac{\sigma}{2}+\frac{\eta(2\eta-\sigma)}{2(-1-\eta+\sigma)}+k\right)!}{2^k\left(\sigma+\frac{\eta(2\eta-\sigma)}{-1-\eta+\sigma}+2k\right)!}\frac{(z+1)^k}{k!}\hat{P}_k \quad (34)$$

We now turn to the first two cases of Table (6), for the hyperbolic case.

- The first case is when $\sigma = 2\eta$ and $M = 2N + 1$, and the coefficients in the three-term recurrence relation (31) are

$$a_k = 16\gamma k\,(k - N - 1)\,(2k - 1 + 2\eta)$$
$$b_k = -4k\,(\eta + k + 2\gamma) - 2\gamma\,(2\eta + 1) - \eta$$

- The even eigenfunctions are

$$\psi(z) = (z+1)^{\frac{\eta}{2}}\, e^{-\frac{\gamma}{2}z} \sum_{k=0}^{\infty} \frac{(\eta + k)\,!}{2^k\,(2\eta + 2k)\,!} \frac{(z+1)^k}{k\,!} \hat{P}_k$$

- For $\mathbf{N = 0}$, $\hat{P}_0(E) = 1$, and from $\hat{P}_1(E) = 0$ we obtain the eigenvalue

$$\hat{P}_1 = (E - b_0)\,\hat{P}_0 - a_0 \hat{P}_{-1} = E + 2\gamma\,(2\eta + 1) + \eta, \;\Rightarrow\; E = -2\gamma\,(2\eta + 1) - \eta$$

- For $\mathbf{N = 1}$, we have that $\hat{P}_0 = 1$, $\hat{P}_1 = (E + 2\gamma\,(2\eta + 1) + \eta)\,\hat{P}_0$ and $\hat{P}_2 = (E + 2\gamma\,(2\eta + 5) + 5\eta + 4)\,\hat{P}_1 + 16\gamma\,(2\eta + 1)\,\hat{P}_0$
- With the condition $\hat{P}_2 = 0$, we find that the eigenvalues are $E = -3\eta - 6\gamma - 2 - 4\gamma\eta \pm 2\sqrt{(\eta + 1)^2 + 4\gamma\,(\gamma - \eta)}$.

- As for the case with $\mathbf{N} = \mathbf{2}$, we find

$$\hat{P}_1 = E + 2\gamma\,(2\eta + 1) + \eta$$
$$\hat{P}_2 = (E + 2\gamma\,(2\eta + 5) + 5\eta + 4)\,\hat{P}_1 + 32\gamma\,(2\eta + 1)\,\hat{P}_0$$
$$\hat{P}_3 = (E + 2\gamma\,(2\eta + 13) + 13\eta + 36)\,\hat{P}_2 + 448\hat{P}_1$$

Setting $\hat{P}_3 = 0$, we can find the eigenvalues, which coincide with those given in TF1 by polynomial expansion.

- For the odd solutions of the second case, with $\sigma = 2\eta + 1$ and $M = 2N + 2$, we have

$$a_k = 16\gamma k\,(k - N - 1)\,(2k + 2\eta - 1)$$
$$b_k = -4k\,(\eta + 1 + k + 2\gamma) - 2\gamma\,(2\eta + 1) - 3\eta - 1$$

and the eigenfunctions are

$$\psi(z) = (z+1)^{\frac{\eta}{2}}\,(z-1)^{\frac{1}{2}}\,e^{-\frac{\gamma}{2}z}\sum_{k=0}^{\infty}\frac{(\eta + k)\,!}{2^k\,(2\eta + 2k)\,!}\,\frac{(z+1)^k}{k\,!}\,\hat{P}_k$$

- For $\mathbf{N} = \mathbf{0}$, we have $\hat{P}_0 = 1$ and $\hat{P}_1 = E + 2\gamma\,(2\eta + 1) + 3\eta + 1$ therefore, the eigenvalue is $E = -2\gamma\,(2\eta + 1) - 3\eta - 1$.
- For $\mathbf{N} = \mathbf{1}$, we find $\hat{P}_1 = E + 2\gamma\,(2\eta + 1) + 3\eta + 1$ and $\hat{P}_2 = (E + 2\gamma\,(2\eta + 5) + 7\eta + 9)\,(E + 2\gamma\,(2\eta + 1) + 3\eta + 1) + 16\gamma\,(2\eta + 1)$, from which we find the energy eigenvalues $E = -5 - 5\eta - 6\gamma - 4\gamma\eta \pm 2\sqrt{(\eta + 2)^2 + 4\gamma\,(\gamma - \eta + 1)}$.

- Finally, for $\mathbf{N} = \mathbf{2}$, we obtain three eigenvalues as solutions of the equation

$$
\begin{aligned}
&(E + 2\gamma(2\eta+9) + 11\eta + 25)(E + 2\gamma(2\eta+5) + 7\eta + 9)\\
&(E + 2\gamma(2\eta+1) + 3\eta + 1) + 32\gamma(2\eta+3)(E + 2\gamma(2\eta+9) + 11\eta + 25)\\
&+ 32\gamma(2\eta+1)(E + 2\gamma(2\eta+1) + 3\eta + 1) = 0.
\end{aligned}
$$

- We now consider the case of the trigonometric potential function (3)

$$U(x) = -4\gamma^2 \cos^4(x) + 4\gamma(\eta+\gamma+M)\cos^2(x) + \eta(\eta-1)\tan^2(x) \quad (35)$$

where $V_1 = 4\gamma(\eta+\gamma+M)$ and M and the eigenfunctions as given in Table 5.

- We shall use as reference the algebra developed by

$$U(x) = A\sin^2(\sqrt{\nu}x) + B\sin(\sqrt{\nu}x) + C\tan(\sqrt{\nu}x)\sec(\sqrt{\nu}x) + D\sec^2(\sqrt{\nu}x) \quad (36)$$

- Under the change of variable $x \to x - \pi/4$ and $\nu = 4$, we mathcing the equation (36) and (37). Therefore we obtain the valus of cosntants $A = -\gamma^2$, $B = -2\gamma(\eta+M)$ and $C = D = 2\eta(\eta-1)$.
- Using these constants to we obtaion the gauge transformation function

$$\hat{\mu}(z) = (1+z)^{\frac{1}{4}\left(-\eta+\sigma+\frac{\eta(\eta-1)}{\eta-\sigma+1}\right)}(1-z)^{\frac{1}{4}\left(-\eta+\sigma-\frac{\eta(\eta-1)}{\eta-\sigma+1}\right)} e^{\frac{\gamma}{2}z} \quad (37)$$

- We find the gauge Hamiltonian
$$\hat{H}_g(z) = -4J_-^2 + 4J_0^2 + 4\gamma J_+ - 4\left(-\sigma + \eta - N\right) J_0 - 4\left(\gamma + \frac{\eta\left(\eta - 1\right)}{1 + \eta - \sigma}\right) J_- + c_*$$
where
$c_* = N^2 + 2N\left(\sigma - \eta\right) + 4\gamma N + 2\gamma\left(\sigma + 1 - \eta\right) - \eta\left(\eta - 1\right) - \frac{2\gamma\eta(\eta-1)}{1+\eta-\sigma} + \left(\eta - \sigma\right)^2.$
- Then for the eigenfunctions $\psi(z) = \hat{\mu}(z)\hat{\chi}(z)$, we look for the ortogonal polynomials part
$$\hat{\chi}_E(z) = \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{\sigma}{2} + \frac{\eta(2\eta-\sigma)}{2(-1-\eta+\sigma)} + k\right)!}{2^k\left(\sigma + \frac{\eta(2\eta-\sigma)}{-1-\eta+\sigma} + 2k\right)!} \frac{(1-z)^k}{k\,!} \hat{P}_k \qquad (38)$$
with three-term recurrence relation
$$\hat{P}_{k+1} = (E - b_k)\hat{P}_k - a_k \hat{P}_{k-1}, \quad k \geq 0$$
where
$$a_k = 16\gamma k\left(k - N - 1\right)\left(2k - 1 + \sigma - \eta + \frac{\eta\left(\eta - 1\right)}{-1 - \eta + \sigma}\right)$$
$$b_k = -4k\left(-\sigma + \eta - k - 2\gamma\right) + 2\gamma\left(\sigma + 1 - \eta\right) - \eta\left(\eta - 1\right) + \frac{2\gamma\eta\left(\eta - 1\right)}{-1 - \eta + \sigma} + \left(\eta - \sigma\right)^2$$

	M	σ	Trigonometric eigenfunctions
1T	2N + 1	2η	$\Phi_1 = e^{-\gamma \cos^2(x)} \cos^\eta(x) f(x)$
2T	2N + 2	$2\eta + 1$	$\Phi_2 = e^{-\gamma \cos^2(x)} \cos^\eta(x) \sin(x) f(x)$

Table 5: Four transformations which are solutions from the trigonometric potential.

- For even functions in this case, we have that $\sigma = 2\eta$ and $M = 2N + 1$. The coefficients in the three-term recurrence relation are

$$a_k = 16\gamma k\,(k - N - 1)\,(2k - 1 + 2\eta)$$
$$b_k = -4k\,(-\eta - k - 2\gamma) + 2\gamma\,(2\eta + 1) + \eta$$

- Here we solve for the three cases $\mathbf{N} = \mathbf{0}$, $\mathbf{1}$ and $\mathbf{2}$.
- For $\mathbf{N} = \mathbf{0}$, the identity $\hat{P}_1 = 0$ gives $E = 2\gamma\,(2\eta + 1) + \eta$.
- For $\mathbf{N} = \mathbf{1}$, when $\hat{P}_2 = 0$ we obtain $(E - 2\gamma\,(2\eta + 5) - 5\eta - 4)\,(E - 2\gamma\,(2\eta + 1) - \eta) + 16\gamma\,(2\eta + 1) = 0$.
- In the case $\mathbf{N} = \mathbf{2}$, setting

$$\hat{P}_3 = (E - 2\gamma\,(2\eta + 9) - 9\eta - 16)\,(E - 2\gamma\,(2\eta + 5) - 5\eta - 4)\,(E - 2\gamma\,(2\eta + 1) - \eta) + 32\,(2\eta + 1)\,(E - 2\gamma\,(2\eta + 9) - 9\eta - 16) + 32\gamma\,(2\eta + 3)\,(E - 2\gamma\,(2\eta + 1) - \eta)$$

we find the corresponding eigenvalues.

- As for the odd solutions in the trigonometric case, $\sigma = 2\eta + 1$ and $M = 2N + 2$. The three-term recurrence relation coefficients are now

$$a_k = 16\gamma k\,(k - N - 1)\,(2k + 2\eta - 1)$$
$$b_k = 4k\,(\eta + 1 + k + 2\gamma) + 2\gamma\,(2\eta + 1) + 3\eta + 1$$

- For $\mathbf{N} = \mathbf{0}$, the condition $\hat{P}_1 = 0$ gives the eigenvalue $E = 2\gamma\,(2\eta + 1) + 3\eta + 1$.
- For $\mathbf{N} = \mathbf{1}$, the condition $\hat{P}_2 = (E - 2\gamma\,(2\eta + 5) - 7\eta - 9)\,(E - 2\gamma\,(2\eta + 1) - 3\eta - 1) + 16\gamma\,(2\eta + 1) = 0$ gives the desired energy eigenvalues, and for
- $\mathbf{N} = \mathbf{2}$, setting

$$\begin{aligned}\hat{P}_3 = &(E - 2\gamma\,(2\eta + 9) - 11\eta - 25)\,(E - 2\gamma\,(2\eta + 5) - 7\eta - 9)\\ &(E - 2\gamma\,(2\eta + 1) - 3\eta - 1) + 32\gamma\,(2\eta + 1)\,(E - 2\gamma\,(2\eta + 9) - 11\eta - 25)\\ &+ 32\gamma\,(2\eta + 3)\,(E - 2\gamma\,(2\eta + 1) - 3\eta - 1) = 0\end{aligned}$$

we get the energy eigenvalues.

- We have shown here the equivalence of the three prefered algebraic procedures to find the exact solutions of QES potentials, solving exactly three examples for the different even and odd solutions in each case.
- The relations between corresponding parameters in the three aproaches have been shown, in some cases the analytical values are more directly found in one procedure than the other, as was shown for the matching terms procedure in which the hamiltonian has been reduced using the gauge functions.